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Fubini's Theorem

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Summary. Fubini theorem is an essential tool for the analysis of high-dimensional space [8], [2], [3], a theorem about the multiple integral and iterated integral. The author has been working on formalizing Fubini's theorem over the past few years [4], [6] in the Mizar system [7], [1]. As a result, Fubini's theorem (30) was proved in complete form by this article.

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1. Preliminaries

From now on X denotes a set.

Now we state the proposition:

(1) Let us consider a subset A of X, and an X-defined binary relation f. Then $f \restriction A^{\mathrm{c}} = f \restriction (\mathrm{dom}\, f \setminus A)$.

Let us consider a partial function f from X to $\overline{\mathbb{R}}$. Now we state the propositions:

(2) GTE-dom$(f, +\infty) =$ EQ-dom$(f, +\infty)$.

(3) LEQ-dom$(f, -\infty) =$ EQ-dom$(f, -\infty)$.

(4) Let us consider a partial function f from X to $\overline{\mathbb{R}}$, and an extended real e. Then GTE-dom(f, e) misses LE-dom(f, e).

(5) Let us consider a partial function f from X to $\overline{\mathbb{R}}$. Then $\mathrm{dom}\, f =$ (EQ-dom $(f, -\infty) \cup$ GT-dom$(f, -\infty) \cap$ LE-dom$(f, +\infty)) \cup$ EQ-dom$(f, +\infty)$.

In the sequel X, X_1, X_2 denote non empty sets.

(6) Let us consider a partial function f from X to $\overline{\mathbb{R}}$, and an element x of X. Then

(i) $(\max_+(f))(x) \leqslant |f|(x)$, and

(ii) $(\max_-(f))(x) \leqslant |f|(x)$.

(7) Let us consider a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, an element x of X_1, and an element y of X_2. Then

(i) $\mathrm{ProjPMap1}(|f|, x) = |\mathrm{ProjPMap1}(f, x)|$, and

(ii) $\mathrm{ProjPMap2}(|f|, y) = |\mathrm{ProjPMap2}(f, y)|$.

2. Markov's Inequality

From now on S denotes a σ-field of subsets of X, S_1 denotes a σ-field of subsets of X_1, S_2 denotes a σ-field of subsets of X_2, M denotes a σ-measure on S, M_1 denotes a σ-measure on S_1, and M_2 denotes a σ-measure on S_2.

Let X be a non empty set, S be a σ-field of subsets of X, and E be an element of S. One can verify that there exists a partial function from X to $\overline{\mathbb{R}}$ which is E-measurable.

Now we state the proposition:

(8) Let us consider an element E of S, and an E-measurable partial function f from X to $\overline{\mathbb{R}}$. Suppose $\mathrm{dom}\, f = E$.
Then $\text{EQ-dom}(f, +\infty)$, $\text{EQ-dom}(f, -\infty) \in S$.

Let us consider an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$ and an E-measurable partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Now we state the propositions:

(9) Suppose M_1 is σ-finite and M_2 is σ-finite and $\mathrm{dom}\, f = E$. Then

(i) $\int \mathrm{Integral2}(M_2, |f|)\, \mathrm{d}M_1 = \int |f|\, \mathrm{d}\, \mathrm{ProdMeas}(M_1, M_2)$, and

(ii) $\int \mathrm{Integral1}(M_1, |f|)\, \mathrm{d}M_2 = \int |f|\, \mathrm{d}\, \mathrm{ProdMeas}(M_1, M_2)$.

(10) Suppose M_1 is σ-finite and M_2 is σ-finite and $E = \mathrm{dom}\, f$. Then f is integrable on $\mathrm{ProdMeas}(M_1, M_2)$ if and only if $\int \mathrm{Integral1}(M_1, |f|)\, \mathrm{d}M_2 < +\infty$.

(11) Suppose M_1 is σ-finite and M_2 is σ-finite and $E = \mathrm{dom}\, f$. Then f is integrable on $\mathrm{ProdMeas}(M_1, M_2)$ if and only if $\int \mathrm{Integral2}(M_2, |f|)\, \mathrm{d}M_1 < +\infty$.

(12) Let us consider an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, an element U of S_1, and an E-measurable partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Suppose M_2 is σ-finite and $E = \mathrm{dom}\, f$. Then $\mathrm{Integral2}(M_2, |f|)$ is U-measurable.

(13) Let us consider an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, an element V of S_2, and an E-measurable partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Suppose M_1 is σ-finite and $E = \mathrm{dom}\, f$. Then $\mathrm{Integral1}(M_1, |f|)$ is V-measurable.

Let us consider a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Now we state the propositions:

(14) Suppose M_2 is σ-finite and f is integrable on $\mathrm{ProdMeas}(M_1, M_2)$. Then

(i) $\int \max_+(\mathrm{Integral2}(M_2, |f|))\, \mathrm{d}M_1 = \int \mathrm{Integral2}(M_2, |f|)\, \mathrm{d}M_1$, and

(ii) $\int \max_-(\mathrm{Integral2}(M_2, |f|))\, \mathrm{d}M_1 = 0$.

The theorem is a consequence of (12).

(15) Suppose M_1 is σ-finite and f is integrable on $\mathrm{ProdMeas}(M_1, M_2)$. Then

(i) $\int \max_+(\mathrm{Integral1}(M_1, |f|))\, \mathrm{d}M_2 = \int \mathrm{Integral1}(M_1, |f|)\, \mathrm{d}M_2$, and

(ii) $\int \max_-(\mathrm{Integral1}(M_1, |f|))\, \mathrm{d}M_2 = 0$.

The theorem is a consequence of (13).

(16) Markov's inequality:

Let us consider an element E of S, an E-measurable partial function f from X to $\overline{\mathbb{R}}$, and an extended real e. Suppose $\mathrm{dom}\, f = E$ and f is non-negative and $e \geqslant 0$. Then $e \cdot M(\text{GTE-dom}(f, e)) \leqslant \int f\, \mathrm{d}M$.

Proof: $\text{GTE-dom}(f, +\infty) = \text{EQ-dom}(f, +\infty)$. Reconsider $E_3 = \text{GTE-dom}(f, e)$ as an element of S. For every element x of X such that $x \in \mathrm{dom}(\chi_{e,E_3,X} \restriction E_3)$ holds $(\chi_{e,E_3,X} \restriction E_3)(x) \leqslant (f \restriction E_3)(x)$. □

3. Fubini's Theorem

Now we state the propositions:

(17) Let us consider partial functions f, g from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M and g is integrable on M. Then

(i) $\int f + g\, \mathrm{d}M = \int f \restriction (\mathrm{dom}\, f \cap \mathrm{dom}\, g)\, \mathrm{d}M + \int g \restriction (\mathrm{dom}\, f \cap \mathrm{dom}\, g)\, \mathrm{d}M$, and

(ii) $\int f - g\, \mathrm{d}M = \int f \restriction (\mathrm{dom}\, f \cap \mathrm{dom}\, g)\, \mathrm{d}M - \int g \restriction (\mathrm{dom}\, f \cap \mathrm{dom}\, g)\, \mathrm{d}M$.

(18) Let us consider a partial function f from X to $\overline{\mathbb{R}}$. Then f is integrable on M if and only if $\max_+(f)$ is integrable on M and $\max_-(f)$ is integrable on M.

(19) Let us consider elements A, B of S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose $B \subseteq A$ and $f \restriction A$ is B-measurable. Then f is B-measurable.

Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. We say that f is integrable a.e. w.r.t. M if and only if

(Def. 1) there exists an element A of S such that $M(A) = 0$ and $A \subseteq \operatorname{dom} f$ and $f \restriction A^{\mathrm{c}}$ is integrable on M.

Let us consider a partial function f from X to $\overline{\mathbb{R}}$. Now we state the propositions:

(20) If f is integrable a.e. w.r.t. M, then $\operatorname{dom} f \in S$.

(21) If f is integrable on M, then f is integrable a.e. w.r.t. M. The theorem is a consequence of (1).

Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and f be a partial function from X to $\overline{\mathbb{R}}$. We say that f is finite M-a.e. if and only if

(Def. 2) there exists an element A of S such that $M(A) = 0$ and $A \subseteq \operatorname{dom} f$ and $f \restriction A^{\mathrm{c}}$ is a partial function from X to $\mathbb{R}$.

Now we state the propositions:

(22) Let us consider an element E of S, and an E-measurable partial function f from X to $\overline{\mathbb{R}}$. Suppose $\operatorname{dom} f = E$. Then f is finite M-a.e. if and only if $M(\text{EQ-dom}(f, +\infty) \cup \text{EQ-dom}(f, -\infty)) = 0$. The theorem is a consequence of (8).

(23) Let us consider a partial function f from X to $\overline{\mathbb{R}}$. Suppose f is integrable on M. Then

(i) $M(\text{EQ-dom}(f, +\infty)) = 0$, and

(ii) $M(\text{EQ-dom}(f, -\infty)) = 0$, and

(iii) f is finite M-a.e., and

(iv) for every real number r such that $r > 0$ holds $M(\text{GTE-dom}(|f|, r)) < +\infty$.

The theorem is a consequence of (16).

(24) Let us consider a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Suppose M_1 is σ-finite and M_2 is σ-finite and f is integrable on $\text{ProdMeas}(M_1, M_2)$. Then

(i) $\text{Integral1}(M_1, \max_+(f))$ is integrable on M_2, and

(ii) $\text{Integral2}(M_2, \max_+(f))$ is integrable on M_1, and

(iii) $\text{Integral1}(M_1, \max_-(f))$ is integrable on M_2, and

(iv) $\text{Integral2}(M_2, \max_-(f))$ is integrable on M_1, and

(v) $\text{Integral1}(M_1, |f|)$ is integrable on M_2, and

(vi) $\text{Integral2}(M_2, |f|)$ is integrable on M_1.

(25) Let us consider an element E of S, and an E-measurable partial function f from X to $\overline{\mathbb{R}}$. Suppose $\mathrm{dom}\, f \subseteq E$ and f is integrable a.e. w.r.t. M. Then f is integrable on M. The theorem is a consequence of (20) and (1).

(26) Let us consider an element A of S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose $M(A) = 0$ and $A \subseteq \mathrm{dom}\, f$ and $f{\restriction}A^{\mathrm{c}}$ is integrable on M. Then there exists a partial function g from X to $\overline{\mathbb{R}}$ such that

(i) $\mathrm{dom}\, g = \mathrm{dom}\, f$, and

(ii) $f{\restriction}A^{\mathrm{c}} = g{\restriction}A^{\mathrm{c}}$, and

(iii) g is integrable on M, and

(iv) $\int f{\restriction}A^{\mathrm{c}}\, \mathrm{d}M = \int g\, \mathrm{d}M$.

PROOF: Consider B being an element of S such that $B = \mathrm{dom}(f{\restriction}A^{\mathrm{c}})$ and $f{\restriction}A^{\mathrm{c}}$ is B-measurable. $f{\restriction}A^{\mathrm{c}} = f{\restriction}(\mathrm{dom}\, f \setminus A)$. Define $\mathcal{C}[\text{object}] \equiv \$_1 \in A$. Define $\mathcal{F}(\text{object}) = +\infty$. Define $\mathcal{G}(\text{object}) = f(\$_1)$. Consider g being a function such that $\mathrm{dom}\, g = \mathrm{dom}\, f$ and for every object x such that $x \in \mathrm{dom}\, f$ holds if $\mathcal{C}[x]$, then $g(x) = \mathcal{F}(x)$ and if not $\mathcal{C}[x]$, then $g(x) = \mathcal{G}(x)$. For every real number r, $(A \cup B) \cap \text{LE-dom}(g, r) \in S$. $\int f{\restriction}A^{\mathrm{c}}\, \mathrm{d}M = \int g{\restriction}(\mathrm{dom}\, g \setminus A)\, \mathrm{d}M$. □

(27) Let us consider a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Suppose M_1 is σ-finite and M_2 is σ-finite and f is integrable on $\mathrm{ProdMeas}(M_1, M_2)$. Then

(i) $\int f\, \mathrm{d}\, \mathrm{ProdMeas}(M_1, M_2) =$
$\int \mathrm{Integral1}(M_1, \max_+(f))\, \mathrm{d}M_2 - \int \mathrm{Integral1}(M_1, \max_-(f))\, \mathrm{d}M_2$, and

(ii) $\int f\, \mathrm{d}\, \mathrm{ProdMeas}(M_1, M_2) =$
$\int \mathrm{Integral2}(M_2, \max_+(f))\, \mathrm{d}M_1 - \int \mathrm{Integral2}(M_2, \max_-(f))\, \mathrm{d}M_1$.

(28) Let us consider an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and an element y of X_2. Then

(i) if $M_1(\mathrm{MeasurableYsection}(E, y)) \neq 0$, then
$(\mathrm{Integral1}(M_1, \overline{\chi}_{E, X_1 \times X_2}))(y) = +\infty$, and

(ii) if $M_1(\mathrm{MeasurableYsection}(E, y)) = 0$, then
$(\mathrm{Integral1}(M_1, \overline{\chi}_{E, X_1 \times X_2}))(y) = 0$.

(29) Let us consider an element E of $\sigma(\mathrm{MeasRect}(S_1, S_2))$, and an element x of X_1. Then

(i) if $M_2(\mathrm{MeasurableXsection}(E, x)) \neq 0$, then
$(\mathrm{Integral2}(M_2, \overline{\chi}_{E, X_1 \times X_2}))(x) = +\infty$, and

(ii) if $M_2(\mathrm{MeasurableXsection}(E, x)) = 0$, then
$(\mathrm{Integral2}(M_2, \overline{\chi}_{E, X_1 \times X_2}))(x) = 0$.

(30) Fubini's theorem:

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and an element S_3 of S_1. Suppose M_1 is σ-finite and M_2 is σ-finite and f is integrable on ProdMeas(M_1, M_2) and $X_1 = S_3$. Then there exists an element U of S_1 such that

(i) $M_1(U) = 0$, and

(ii) for every element x of X_1 such that $x \in U^{\rm c}$ holds ProjPMap1(f, x) is integrable on M_2, and

(iii) Integral2$(M_2, |f|) {\restriction} U^{\rm c}$ is a partial function from X_1 to $\mathbb{R}$, and

(iv) Integral2(M_2, f) is $(S_3 \setminus U)$-measurable, and

(v) Integral2$(M_2, f) {\restriction} U^{\rm c}$ is integrable on M_1, and

(vi) Integral2$(M_2, f) {\restriction} U^{\rm c} \in$ the L^1 functions of M_1, and

(vii) there exists a function g from X_1 into $\overline{\mathbb{R}}$ such that g is integrable on M_1 and $g {\restriction} U^{\rm c} =$ Integral2$(M_2, f) {\restriction} U^{\rm c}$ and $\int f \, {\rm d}\,$ProdMeas$(M_1, M_2) = \int g \, {\rm d}M_1$.

Proof: Consider A being an element of σ(MeasRect(S_1, S_2)) such that $A = {\rm dom}\, f$ and f is A-measurable. Integral2$(M_2, |f|)$ is integrable on M_1 and Integral2$(M_2, \max_+(f))$ is integrable on M_1 and Integral2$(M_2, \max_-(f))$ is integrable on M_1. Integral2$(M_2, |f|)$ is finite M_1-a.e.. Consider U being an element of S_1 such that $M_1(U) = 0$ and Integral2$(M_2, |f|) {\restriction} U^{\rm c}$ is a partial function from X_1 to $\mathbb{R}$. For every element x of X_1 such that $x \in U^{\rm c}$ holds ProjPMap1(f, x) is integrable on M_2. Consider g_1 being a partial function from X_1 to $\overline{\mathbb{R}}$ such that ${\rm dom}\, g_1 = {\rm dom}$(Integral2$(M_2, \max_+(f))$) and $g_1 {\restriction} U^{\rm c} =$ Integral2$(M_2, \max_+(f)) {\restriction} U^{\rm c}$ and g_1 is integrable on M_1 and $\int g_1 \, {\rm d}M_1 = \int$ Integral2$(M_2, \max_+(f)) {\restriction} U^{\rm c} \, {\rm d}M_1$.

Consider g_2 being a partial function from X_1 to $\overline{\mathbb{R}}$ such that ${\rm dom}\, g_2 = {\rm dom}$(Integral2$(M_2, \max_-(f))$) and $g_2 {\restriction} U^{\rm c} =$ Integral2$(M_2, \max_-(f)) {\restriction} U^{\rm c}$ and g_2 is integrable on M_1 and $\int g_2 \, {\rm d}M_1 = \int$ Integral2$(M_2, \max_-(f)) {\restriction} U^{\rm c} \, {\rm d}M_1$. Consider g being a partial function from X_1 to $\overline{\mathbb{R}}$ such that ${\rm dom}\, g = {\rm dom}$(Integral2$(M_2, f)$) and $g {\restriction} U^{\rm c} =$ Integral2$(M_2, f) {\restriction} U^{\rm c}$ and g is integrable on M_1 and $\int g \, {\rm d}M_1 = \int$ Integral2$(M_2, f) {\restriction} U^{\rm c} \, {\rm d}M_1$. $\int f \, {\rm d}\,$ProdMeas(M_1, M_2) $= \int g {\restriction} U^{\rm c} \, {\rm d}M_1$. □

(31) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$, and an element S_4 of S_2. Suppose M_1 is σ-finite and M_2 is σ-finite and f is integrable on ProdMeas(M_1, M_2) and $X_2 = S_4$. Then there exists an element V of S_2 such that

(i) $M_2(V) = 0$, and

(ii) for every element y of X_2 such that $y \in V^c$ holds ProjPMap2(f, y) is integrable on M_1, and

(iii) Integral1$(M_1, |f|) \upharpoonright V^c$ is a partial function from X_2 to $\mathbb{R}$, and

(iv) Integral1(M_1, f) is $(S_4 \setminus V)$-measurable, and

(v) Integral1$(M_1, f) \upharpoonright V^c$ is integrable on M_2, and

(vi) Integral1$(M_1, f) \upharpoonright V^c \in$ the L^1 functions of M_2, and

(vii) there exists a function g from X_2 into $\overline{\mathbb{R}}$ such that g is integrable on M_2 and $g \upharpoonright V^c =$ Integral1$(M_1, f) \upharpoonright V^c$ and $\int f \, \mathrm{d}\,\mathrm{ProdMeas}(M_1, M_2) = \int g \, \mathrm{d}M_2$.

PROOF: Consider A being an element of σ(MeasRect(S_1, S_2)) such that $A = \mathrm{dom}\, f$ and f is A-measurable. Integral1$(M_1, |f|)$ is integrable on M_2 and Integral1$(M_1, \max_+(f))$ is integrable on M_2 and Integral1$(M_1, \max_-(f))$ is integrable on M_2. Integral1$(M_1, |f|)$ is finite M_2-a.e.. Consider V being an element of S_2 such that $M_2(V) = 0$ and Integral1$(M_1, |f|) \upharpoonright V^c$ is a partial function from X_2 to $\mathbb{R}$. For every element y of X_2 such that $y \in V^c$ holds ProjPMap2(f, y) is integrable on M_1 by (7), [5, (31)]. Consider g_1 being a partial function from X_2 to $\overline{\mathbb{R}}$ such that $\mathrm{dom}\, g_1 =$ dom(Integral1$(M_1, \max_+(f))$) and $g_1 \upharpoonright V^c =$ Integral1$(M_1, \max_+(f)) \upharpoonright V^c$ and g_1 is integrable on M_2 and $\int g_1 \, \mathrm{d}M_2 = \int$ Integral1$(M_1, \max_+(f)) \upharpoonright V^c$ $\mathrm{d}M_2$.

Consider g_2 being a partial function from X_2 to $\overline{\mathbb{R}}$ such that $\mathrm{dom}\, g_2 =$ dom(Integral1$(M_1, \max_-(f))$) and $g_2 \upharpoonright V^c =$ Integral1$(M_1, \max_-(f)) \upharpoonright V^c$ and g_2 is integrable on M_2 and $\int g_2 \, \mathrm{d}M_2 = \int$ Integral1$(M_1, \max_-(f)) \upharpoonright V^c$ $\mathrm{d}M_2$. Consider g being a partial function from X_2 to $\overline{\mathbb{R}}$ such that $\mathrm{dom}\, g =$ dom(Integral1(M_1, f)) and $g \upharpoonright V^c =$ Integral1$(M_1, f) \upharpoonright V^c$ and g is integrable on M_2 and $\int g \, \mathrm{d}M_2 = \int$ Integral1$(M_1, f) \upharpoonright V^c \, \mathrm{d}M_2$. $\int f \, \mathrm{d}\,\mathrm{ProdMeas}(M_1, M_2)$ $= \int g \upharpoonright V^c \, \mathrm{d}M_2$. □

Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and a partial function f from $X_1 \times X_2$ to $\overline{\mathbb{R}}$. Now we state the propositions:

(32) Suppose M_1 is σ-finite and M_2 is σ-finite and f is integrable on ProdMeas (M_1, M_2) and for every element x of X_1, (Integral2$(M_2, |f|)$)$(x) < +\infty$. Then

(i) for every element x of X_1, ProjPMap1(f, x) is integrable on M_2, and

(ii) for every element U of S_1, Integral2(M_2, f) is U-measurable, and

(iii) Integral2(M_2, f) is integrable on M_1, and

(iv) $\int f \, \mathrm{d}\, \mathrm{ProdMeas}(M_1, M_2) = \int \mathrm{Integral2}(M_2, f) \, \mathrm{d}M_1$, and

(v) $\mathrm{Integral2}(M_2, f) \in$ the L^1 functions of M_1.

The theorem is a consequence of (7), (24), (6), and (17).

(33) Suppose M_1 is σ-finite and M_2 is σ-finite and f is integrable on ProdMeas (M_1, M_2) and for every element y of X_2, $(\mathrm{Integral1}(M_1, |f|))(y) < +\infty$. Then

(i) for every element y of X_2, $\mathrm{ProjPMap2}(f, y)$ is integrable on M_1, and

(ii) for every element V of S_2, $\mathrm{Integral1}(M_1, f)$ is V-measurable, and

(iii) $\mathrm{Integral1}(M_1, f)$ is integrable on M_2, and

(iv) $\int f \, \mathrm{d}\, \mathrm{ProdMeas}(M_1, M_2) = \int \mathrm{Integral1}(M_1, f) \, \mathrm{d}M_2$, and

(v) $\mathrm{Integral1}(M_1, f) \in$ the L^1 functions of M_2.

The theorem is a consequence of (7), (24), (6), and (17).

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