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Fundamental Properties of Fuzzy Implications

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Summary. In the article we continue in the Mizar system [8], [2] the formalization of fuzzy implications according to the monograph of Baczyński and Jayaram *"Fuzzy Implications"* [1]. We develop a framework of Mizar attributes allowing us for a smooth proving of basic properties of these fuzzy connectives [9]. We also give a set of theorems about the ordering of nine fundamental implications: Łukasiewicz (I_{LK}), Gödel (I_{GD}), Reichenbach (I_{RC}), Kleene-Dienes (I_{KD}), Goguen (I_{GG}), Rescher (I_{RS}), Yager (I_{YG}), Weber (I_{WB}), and Fodor (I_{FD}).

This work is a continuation of the development of fuzzy sets in Mizar [6]; it could be used to give a variety of more general operations on fuzzy sets [13]. The formalization follows [10], [5], and [4].

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0. Introduction

There are two fundamental aims of this Mizar article: first of all, I wanted to introduce in the Mizar Mathematical Library how nine basic fuzzy implications formally defined in [4] are ordered – and this result is given in Section 2 as a formal counterpart of Example 1.1.6, p. 3 of [1].

On the other hand, in the final section I prove the formal characterization of fundamental fuzzy implications in terms of four elementary properties [12] expressed in Table 1.4 of [1], p. 10 (note the absence of the continuity of the operators in our version of this presentation). Here

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- (NP) – the left neutrality property,
- (EP) – the exchange principle,
- (IP) – the identity principle,
- (OP) – the ordering property.

Actually, this is the part of Example 1.3.2, p. 9 from [1]:

Fuzzy implication	(NP)	(EP)	(IP)	(OP)
I_{LK}	+	+	+	+
I_{GD}	+	+	+	+
I_{RC}	+	+	−	−
I_{KD}	+	+	−	−
I_{GG}	+	+	+	+
I_{RS}	−	−	+	+
I_{YG}	+	+	−	−
I_{WB}	+	+	+	−
I_{FD}	+	+	+	+

Additionally, Section 4 contains registrations of clusters of adjectives allowing for further work in more automated framework within fuzzy sets [3] – this is the Mizar version of Lemma 1.3.3 and 1.3.4 from [1]. Such automatization can be especially useful in the hybridization of fuzzy and rough approaches [7].

1. Preliminaries

We introduce the notation I_{LK} as a synonym of the Łukasiewicz implication and I_{GD} as a synonym of the Gödel implication. We introduce I_{RC} as a synonym of the Reichenbach implication and I_{KD} as a synonym of the Kleene-Dienes implication.

We introduce I_{GG} as a synonym of the Goguen implication and I_{RS} as a synonym of the Rescher implication. We introduce I_{YG} as a synonym of the Yager implication and I_{WB} as a synonym of the Weber implication and I_{FD} as a synonym of the Fodor implication.

From now on x, y denote elements of $[0,1]$. Now we state the propositions:

(1) $\square^1 = (\mathrm{AffineMap}(1,0)) \restriction]0,+\infty[$.

Proof: Set $f = \square^1$. Set $g = (\mathrm{AffineMap}(1,0)) \restriction]0,+\infty[$. For every object x such that $x \in \mathrm{dom}\, f$ holds $f(x) = g(x)$. $\square$

(2) Let us consider real numbers a, b. Then

(i) AffineMap(a, b) is differentiable on $\mathbb{R}$, and

(ii) for every real number x, $(\text{AffineMap}(a, b))'(x) = a$.

(3) If $0 < x < 1$ and $0 < y < 1$, then $(\square^x + (\text{AffineMap}(-x, x-1)))\restriction]0,1[$ is increasing.

PROOF: Set $f_1 = \square^x$. Set $f_2 = \text{AffineMap}(-x, x-1)$. Reconsider $Y =]0,1[$ as an open subset of $\mathbb{R}$. Set $f = f_1 + f_2$. Set $A =]0, +\infty[$. f_2 is differentiable on A. $f_1 \restriction A$ is differentiable on A. f_2 is differentiable on Y. For every real number y such that $y \in Y$ holds $0 < f'(y)$ by [11, (21)], (2). $\square$

(4) Let us consider a real number u. Suppose $u \in]0, 1]$.
Then $(\square^x + (\text{AffineMap}(-x, x-1)))(u) = u^x - 1 + x - x \cdot u$.

2. The Ordering of Fuzzy Implications

Now we state the propositions:

(5) (i) if $x \leqslant y$, then $(I_{\text{LK}})(x, y) = 1$, and

(ii) if $x > y$, then $(I_{\text{LK}})(x, y) = 1 - x + y$.

(6) (i) if $x = 0$, then $(I_{\text{GG}})(x, y) = 1$, and

(ii) if $x > 0$, then $(I_{\text{GG}})(x, y) = \min(1, \frac{y}{x})$.

(7) $I_{\text{KD}} \leqslant I_{\text{RC}} \leqslant I_{\text{LK}} \leqslant I_{\text{WB}}$.

(8) $I_{\text{RS}} \leqslant I_{\text{GD}} \leqslant I_{\text{GG}} \leqslant I_{\text{LK}} \leqslant I_{\text{WB}}$.

(9) $I_{\text{RC}} \leqslant I_{\text{LK}} \leqslant I_{\text{WB}}$.

(10) $I_{\text{KD}} \leqslant I_{\text{FD}} \leqslant I_{\text{LK}} \leqslant I_{\text{WB}}$.

(11) $I_{\text{RS}} \leqslant I_{\text{GD}} \leqslant I_{\text{FD}} \leqslant I_{\text{LK}} \leqslant I_{\text{WB}}$.

3. Additional Properties of Fuzzy Implications

Let I be a binary operation on $[0, 1]$. We say that I satisfies (NP) if and only if

(Def. 1) for every element y of $[0, 1]$, $I(1, y) = y$.

We say that I satisfies (EP) if and only if

(Def. 2) for every elements x, y, z of $[0, 1]$, $I(x, I(y, z)) = I(y, I(x, z))$.

We say that I satisfies (IP) if and only if

(Def. 3) for every element x of $[0, 1]$, $I(x, x) = 1$.

We say that I satisfies (OP) if and only if

(Def. 4) for every elements x, y of $[0, 1]$, $I(x, y) = 1$ iff $x \leqslant y$.

In the sequel I denotes a binary operation on $[0,1]$.

Let I be a binary operation on $[0,1]$. We introduce the notation I satisfies (NC) as a synonym of I is 01-dominant and I satisfies (I1) as a synonym of I is antitone w.r.t. 1st coordinate.

We introduce I satisfies (I2) as a synonym of I is isotone w.r.t. 2nd coordinate and I satisfies (I3) as a synonym of I is 00-dominant and I satisfies (I4) as a synonym of I is 11-dominant and I satisfies (I5) as a synonym of I is 10-weak.

4. Dependencies between Chosen Properties

Now we state the proposition:

(12) If I satisfies (LB), then I satisfies (I3) and (NC).

One can verify that every binary operation on $[0,1]$ which satisfies (LB) satisfies also (I3) and (NC).

Now we state the proposition:

(13) If I satisfies (RB), then I satisfies (I4) and (NC).

One can check that every binary operation on $[0,1]$ which satisfies (RB) satisfies also (I4) and (NC).

Now we state the proposition:

(14) If I satisfies (NP), then I satisfies (I4) and (I5).

Note that every binary operation on $[0,1]$ which satisfies (NP) satisfies also (I4) and (I5).

Now we state the proposition:

(15) If I satisfies (IP), then I satisfies (I3) and (I4).

Let us note that every binary operation on $[0,1]$ which satisfies (IP) satisfies also (I3) and (I4).

Now we state the proposition:

(16) If I satisfies (OP), then I satisfies (I3), (I4), (NC), (LB), (RB), and (IP).

One can verify that every binary operation on $[0,1]$ which satisfies (OP) satisfies also (I3), (I4), (NC), (LB), (RB), and (IP).

Now we state the proposition:

(17) If I satisfies (EP) and (OP), then I satisfies (I1), (I3), (I4), (I5), (LB), (RB), (NC), (NP), and (IP).

One can verify that every binary operation on $[0,1]$ which satisfies (EP) and (OP) satisfies also (I1), (I5), and (NP).

5. Properties of Nine Classical Fuzzy Implications

Let us note that I_{LK} satisfies (NP), (EP), (IP), and (OP).

I_{GD} satisfies (NP), (EP), (IP), and (OP).

I_{RC} satisfies (NP) and (EP) but does not satisfy (IP) and (OP).

I_{KD} satisfies (NP) and (EP) but does not satisfy (IP) and (OP).

I_{GG} satisfies (NP), (EP), (IP), and (OP).

Let us note that I_{RS} satisfies (IP) and (OP) but does not satisfy (NP) and (EP).

I_{YG} satisfies (NP) and (EP) but does not satisfy (IP) and (OP).

I_{WB} satisfies (NP), (EP), and (IP) but does not satisfy (OP).

I_{FD} satisfies (NP), (EP), (IP), and (OP).

I_0 satisfies (EP) but does not satisfy (NP), (IP), and (OP).

I_1 satisfies (EP) and (IP) but does not satisfy (NP) and (OP).

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