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About Supergraphs. Part I

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Summary. Drawing a finite graph is usually done by a finite sequence of the following three operations.

1. Draw a vertex of the graph.
2. Draw an edge between two vertices of the graph.
3. Draw an edge starting from a vertex of the graph and immediately draw a vertex at the other end of it.

By this procedure any finite graph can be constructed. This property of graphs is so obvious that the author of this article has yet to find a reference where it is mentioned explicitly. In introductionary books (like [10], [5], [9]) as well as in advanced ones (like [4]), after the initial definition of graphs the examples are usually given by graphical representations rather than explicit set theoretic descriptions, assuming a mutual understanding how the representation is to be translated into a description fitting the definition. However, Mizar [2], [3] does not possess this innate ability of humans to translate pictures into graphs. Therefore, if one wants to create graphs in Mizar without directly providing a set theoretic description (i.e. using the `createGraph` functor), a rigorous approach to the constructing operations is required.

In this paper supergraphs are defined as an inverse mode to subgraphs as given in [8]. The three graph construction operations are defined as modes extending `Supergraph` similar to how the various remove operations were introduced as submodes of `Subgraph` in [8]. Many theorems are proven that describe how graph properties are transferred to special supergraphs. In particular, to prove that disconnected graphs cannot become connected by adding an edge between two vertices that lie in the same component, the theory of replacing a part of a walk with another walk is introduced in the preliminaries.

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1. General Preliminaries

Let us consider an even integer n and an odd integer m. Now we state the propositions:

(1) If $n \leqslant m$, then $n+1 \leqslant m$.

(2) If $m \leqslant n$, then $m+1 \leqslant n$.

(3) Let us consider natural numbers i, j. If $i > i -' 1 + j$, then $j = 0$.

(4) Let us consider finite sequences f, g, and a natural number i. Suppose $i \leqslant \operatorname{len} f$ and $\operatorname{mid}(f, i, i -' 1 + \operatorname{len} g) = g$. Then $i -' 1 + \operatorname{len} g \leqslant \operatorname{len} f$. The theorem is a consequence of (3).

Let us consider a finite sequence p and a natural number n. Now we state the propositions:

(5) If $n \in \operatorname{dom} p$ and $n+1 \leqslant \operatorname{len} p$, then $\operatorname{mid}(p, n, n+1) = \langle p(n), p(n+1)\rangle$.

(6) If $n \in \operatorname{dom} p$ and $n+2 \leqslant \operatorname{len} p$, then $\operatorname{mid}(p, n, n+2) = \langle p(n), p(n+1), p(n+2)\rangle$. The theorem is a consequence of (5).

(7) Let us consider a non empty set D, finite sequences f, g of elements of D, and a natural number n. Suppose g is a substring of f. Then

   (i) $\operatorname{len} g = 0$, or

   (ii) $1 \leqslant n -' 1 + \operatorname{len} g \leqslant \operatorname{len} f$ and $n \leqslant n -' 1 + \operatorname{len} g$.

   The theorem is a consequence of (4).

Let D be a non empty set, f, g be finite sequences of elements of D, and n be a natural number. We say that g is an odd substring of f not starting before n if and only if

(Def. 1) if $\operatorname{len} g > 0$, then there exists an odd natural number i such that $n \leqslant i \leqslant \operatorname{len} f$ and $\operatorname{mid}(f, i, i -' 1 + \operatorname{len} g) = g$.

We say that g is an even substring of f not starting before n if and only if

(Def. 2) if $\operatorname{len} g > 0$, then there exists an even natural number i such that $n \leqslant i \leqslant \operatorname{len} f$ and $\operatorname{mid}(f, i, i -' 1 + \operatorname{len} g) = g$.

Let us consider a non empty set D, finite sequences f, g of elements of D, and a natural number n. Now we state the propositions:

(8) If g is an odd substring of f not starting before n, then g is a substring of f.

(9) If g is an even substring of f not starting before n, then g is a substring of f.

(10) Let us consider a non empty set D, finite sequences f, g of elements of D, and natural numbers n, m. Suppose $m \geqslant n$. Then

(i) if g is an odd substring of f not starting before m, then g is an odd substring of f not starting before n, and

(ii) if g is an even substring of f not starting before m, then g is an even substring of f not starting before n.

(11) Let us consider a non empty set D, and a finite sequence f of elements of D. If $1 \leqslant \operatorname{len} f$, then f is an odd substring of f not starting before 0.

(12) Let us consider a non empty set D, finite sequences f, g of elements of D, and an even element n of $\mathbb{N}$. Suppose g is an odd substring of f not starting before n. Then g is an odd substring of f not starting before $n+1$.

(13) Let us consider a non empty set D, finite sequences f, g of elements of D, and an odd element n of $\mathbb{N}$. Suppose g is an even substring of f not starting before n. Then g is an even substring of f not starting before $n+1$.

(14) Let us consider a non empty set D, and finite sequences f, g of elements of D. Suppose g is an odd substring of f not starting before 0. Then g is an odd substring of f not starting before 1. The theorem is a consequence of (12).

2. Graph Preliminaries

Let G be a non-directed-multi graph. Observe that every subgraph of G is non-directed-multi.

(15) Every graph is a subgraph of G induced by the vertices of G.

(16) Let us consider graphs G_1, G_3, sets V, E, and a subgraph G_2 of G_1 induced by V and E. If $G_2 \approx G_3$, then G_3 is a subgraph of G_1 induced by V and E.

(17) Let us consider a graph G, a set X, and objects e, y. Suppose e joins a vertex from X and a vertex from $\{y\}$ in G. Then there exists an object x such that

(i) $x \in X$, and

(ii) e joins x and y in G.

(18) Let us consider a graph G, and a set X. Suppose $X \cap$ (the vertices of G) $= \emptyset$. Then

(i) G.edgesInto$(X) = \emptyset$, and

(ii) G.edgesOutOf$(X) = \emptyset$, and

(iii) G.edgesInOut$(X) = \emptyset$, and

(iv) G.edgesBetween(X) = $\emptyset$.

PROOF: G.edgesInto(X) = $\emptyset$. G.edgesOutOf(X) = $\emptyset$. □

Let us consider a graph G, sets X_1, X_2, and an object y. Now we state the propositions:

(19) If X_1 misses X_2, then G.edgesBetween($X_1, \{y\}$) misses G.edgesBetween ($X_2, \{y\}$). The theorem is a consequence of (17).

(20) G.edgesBetween($X_1 \cup X_2, \{y\}$) = G.edgesBetween($X_1, \{y\}$) $\cup$ G.edgesBetween($X_2, \{y\}$).
PROOF: Set $E_1 = G$.edgesBetween($X_1, \{y\}$). Set $E_2 = G$.edgesBetween(X_2, $\{y\}$). For every object e such that $e \in G$.edgesBetween($X_1 \cup X_2, \{y\}$) holds $e \in E_1 \cup E_2$. □

(21) Let us consider a trivial graph G. Then there exists a vertex v of G such that

(i) the vertices of $G = \{v\}$, and

(ii) the source of G = (the edges of G) $\longmapsto v$, and

(iii) the target of G = (the edges of G) $\longmapsto v$.

PROOF: Consider v being a vertex of G such that the vertices of $G = \{v\}$. For every object e such that $e \in$ dom(the source of G) holds (the source of G)(e) $= v$. For every object e such that $e \in$ dom(the target of G) holds (the target of G)(e) $= v$. □

Let G be a graph. Let us note that every walk of G which is closed, trail-like, and non trivial is also circuit-like and every walk of G which is closed, path-like, and non trivial is also cycle-like.

Let us consider graphs G_1, G_2, a walk W_1 of G_1, and a walk W_2 of G_2. Now we state the propositions:

(22) If $W_1 = W_2$, then if W_1 is trail-like, then W_2 is trail-like.

(23) If $W_1 = W_2$, then if W_1 is path-like, then W_2 is path-like. The theorem is a consequence of (22).

(24) If $W_1 = W_2$, then if W_1 is cycle-like, then W_2 is cycle-like. The theorem is a consequence of (23).

(25) If $W_1 = W_2$, then if W_1 is vertex-distinct, then W_2 is vertex-distinct.

(26) Let us consider a graph G, a walk W of G, and a vertex v of G. If $v \in W$.vertices(), then G.walkOf(v) is a substring of W.

(27) Let us consider a graph G, a walk W of G, and an odd element n of $\mathbb{N}$. Suppose $n + 2 \leqslant \operatorname{len} W$. Then G.walkOf($W(n), W(n+1), W(n+2)$) is an odd substring of W not starting before 0. The theorem is a consequence of (6).

Let us consider a graph G, a walk W of G, and objects u, e, v. Now we state the propositions:

(28) Suppose e joins u and v in G and $e \in W$.edges(). Then

(i) G.walkOf(u, e, v) is an odd substring of W not starting before 0, or

(ii) G.walkOf(v, e, u) is an odd substring of W not starting before 0.

The theorem is a consequence of (27).

(29) If e joins u and v in G and G.walkOf(u, e, v) is an odd substring of W not starting before 0, then $e \in W$.edges() and u, $v \in W$.vertices(). The theorem is a consequence of (14), (8), and (7).

Let G be a graph and W_1, W_2 be walks of G.

The functor W_1.findFirstVertex(W_2) yielding an odd element of $\mathbb{N}$ is defined by

(Def. 3) (i) $it \leqslant \operatorname{len} W_1$ and there exists an even natural number k such that $it = k + 1$ and for every natural number n such that $1 \leqslant n \leqslant \operatorname{len} W_2$ holds $W_1(k + n) = W_2(n)$ and for every even natural number l such that for every natural number n such that $1 \leqslant n \leqslant \operatorname{len} W_2$ holds $W_1(l + n) = W_2(n)$ holds $k \leqslant l$, **if** W_2 is an odd substring of W_1 not starting before 0,

(ii) $it = \operatorname{len} W_1$, **otherwise**.

The functor W_1.findLastVertex(W_2) yielding an odd element of $\mathbb{N}$ is defined by

(Def. 4) (i) $it \leqslant \operatorname{len} W_1$ and there exists an even natural number k such that $it = k + \operatorname{len} W_2$ and for every natural number n such that $1 \leqslant n \leqslant \operatorname{len} W_2$ holds $W_1(k + n) = W_2(n)$ and for every even natural number l such that for every natural number n such that $1 \leqslant n \leqslant \operatorname{len} W_2$ holds $W_1(l + n) = W_2(n)$ holds $k \leqslant l$, **if** W_2 is an odd substring of W_1 not starting before 0,

(ii) $it = \operatorname{len} W_1$, **otherwise**.

Let us consider a graph G and walks W_1, W_2 of G. Now we state the propositions:

(30) Suppose W_2 is an odd substring of W_1 not starting before 0. Then

(i) $W_1(W_1$.findFirstVertex$(W_2)) = W_2$.first(), and

(ii) $W_1(W_1$.findLastVertex$(W_2)) = W_2$.last().

(31) Suppose W_2 is an odd substring of W_1 not starting before 0. Then

(i) $1 \leqslant W_1$.findFirstVertex$(W_2) \leqslant \operatorname{len} W_1$, and

(ii) $1 \leqslant W_1$.findLastVertex$(W_2) \leqslant \operatorname{len} W_1$.

(32) Let us consider a graph G, and a walk W of G. Then

(i) $1 = W.\mathrm{findFirstVertex}(W)$, and

(ii) $W.\mathrm{findLastVertex}(W) = \mathrm{len}\, W$.

The theorem is a consequence of (11).

(33) Let us consider a graph G, and walks W_1, W_2 of G. Suppose W_2 is an odd substring of W_1 not starting before 0. Then $W_1.\mathrm{findFirstVertex}(W_2) \leqslant W_1.\mathrm{findLastVertex}(W_2)$.

Let G be a graph and W_1, W_2, W_3 be walks of G. The functor $W_1.\mathrm{replaceWith}(W_2, W_3)$ yielding a walk of G is defined by the term

$$\text{(Def. 5)} \quad \begin{cases} ((W_1.\mathrm{cut}(1, W_1.\mathrm{findFirstVertex}(W_2))).\mathrm{append}(W_3)).\mathrm{append}((W_1.\mathrm{cut} \\ \quad (W_1.\mathrm{findLastVertex}(W_2), \mathrm{len}\, W_1))), \\ \textbf{if } W_2 \text{ is an odd substring of } W_1 \text{ not starting before 0 and } W_2.\mathrm{first}() \\ \quad = W_3.\mathrm{first}() \text{ and } W_2.\mathrm{last}() = W_3.\mathrm{last}(), W_1, \\ \textbf{otherwise}. \end{cases}$$

Let W_1, W_3 be walks of G and e be an object.

The functor $W_1.\mathrm{replaceEdgeWith}(e, W_3)$ yielding a walk of G is defined by the term

$$\text{(Def. 6)} \quad \begin{cases} W_1.\mathrm{replaceWith}(G.\mathrm{walkOf}(W_3.\mathrm{first}(), e, W_3.\mathrm{last}()), W_3), \\ \textbf{if } e \text{ joins } W_3.\mathrm{first}() \text{ and } W_3.\mathrm{last}() \text{ in } G \text{ and } G.\mathrm{walkOf}(W_3.\mathrm{first}(), e, \\ \quad W_3.\mathrm{last}()) \text{ is an odd substring of } W_1 \text{ not starting before } 0, W_1, \\ \textbf{otherwise}. \end{cases}$$

Let W_1, W_2 be walks of G. The functor $W_1.\mathrm{replaceWithEdge}(W_2, e)$ yielding a walk of G is defined by the term

$$\text{(Def. 7)} \quad \begin{cases} W_1.\mathrm{replaceWith}(W_2, G.\mathrm{walkOf}(W_2.\mathrm{first}(), e, W_2.\mathrm{last}())), \\ \textbf{if } W_2 \text{ is an odd substring of } W_1 \text{ not starting before 0 and } e \text{ joins} \\ \quad W_2.\mathrm{first}() \text{ and } W_2.\mathrm{last}() \text{ in } G, W_1, \\ \textbf{otherwise}. \end{cases}$$

Let us consider a graph G and walks W_1, W_2, W_3 of G. Now we state the propositions:

(34) Suppose W_2 is an odd substring of W_1 not starting before 0 and $W_2.\mathrm{first}() = W_3.\mathrm{first}()$ and $W_2.\mathrm{last}() = W_3.\mathrm{last}()$. Then

(i) $(W_1.\mathrm{cut}(1, W_1.\mathrm{findFirstVertex}(W_2))).\mathrm{first}() = W_1.\mathrm{first}()$, and

(ii) $(W_1.\mathrm{cut}(1, W_1.\mathrm{findFirstVertex}(W_2))).\mathrm{last}() = W_3.\mathrm{first}()$, and

(iii) $((W_1.\mathrm{cut}(1, W_1.\mathrm{findFirstVertex}(W_2))).\mathrm{append}(W_3)).\mathrm{first}() = W_1.\mathrm{first}()$, and

(iv) $((W_1.\mathrm{cut}(1, W_1.\mathrm{findFirstVertex}(W_2))).\mathrm{append}(W_3)).\mathrm{last}() = W_3.\mathrm{last}()$, and

(v) $(W_1.\mathrm{cut}(W_1.\mathrm{findLastVertex}(W_2), \mathrm{len}\, W_1)).\mathrm{first}() = W_3.\mathrm{last}()$, and

(vi) $(W_1.\mathrm{cut}(W_1.\mathrm{findLastVertex}(W_2), \mathrm{len}\, W_1)).\mathrm{last}() = W_1.\mathrm{last}()$.

The theorem is a consequence of (31) and (30).

(35) (i) $W_1.\mathrm{first}() = (W_1.\mathrm{replaceWith}(W_2, W_3)).\mathrm{first}()$, and

(ii) $W_1.\mathrm{last}() = (W_1.\mathrm{replaceWith}(W_2, W_3)).\mathrm{last}()$.

The theorem is a consequence of (34).

(36) Suppose W_2 is an odd substring of W_1 not starting before 0 and $W_2.\mathrm{first}() = W_3.\mathrm{first}()$ and $W_2.\mathrm{last}() = W_3.\mathrm{last}()$. Then $(W_1.\mathrm{replaceWith}(W_2, W_3)).\mathrm{vertices}() = ((W_1.\mathrm{cut}(1, W_1.\mathrm{findFirstVertex}(W_2))).\mathrm{vertices}() \cup W_3.\mathrm{vertices}()) \cup (W_1.\mathrm{cut}(W_1.\mathrm{findLastVertex}(W_2), \mathrm{len}\, W_1)).\mathrm{vertices}()$. The theorem is a consequence of (34).

(37) Suppose W_2 is an odd substring of W_1 not starting before 0 and $W_2.\mathrm{first}() = W_3.\mathrm{first}()$ and $W_2.\mathrm{last}() = W_3.\mathrm{last}()$. Then $(W_1.\mathrm{replaceWith}(W_2, W_3)).\mathrm{edges}() = ((W_1.\mathrm{cut}(1, W_1.\mathrm{findFirstVertex}(W_2))).\mathrm{edges}() \cup W_3.\mathrm{edges}()) \cup (W_1.\mathrm{cut}(W_1.\mathrm{findLastVertex}(W_2), \mathrm{len}\, W_1)).\mathrm{edges}()$. The theorem is a consequence of (34).

(38) Let us consider a graph G, walks W_1, W_3 of G, and an object e. Suppose e joins $W_3.\mathrm{first}()$ and $W_3.\mathrm{last}()$ in G and $G.\mathrm{walkOf}(W_3.\mathrm{first}(), e, W_3.\mathrm{last}())$ is an odd substring of W_1 not starting before 0.
Then $e \in (W_1.\mathrm{replaceEdgeWith}(e, W_3)).\mathrm{edges}()$ if and only if $e \in (W_1.\mathrm{cut}(1, W_1.\mathrm{findFirstVertex}(G.\mathrm{walkOf}(W_3.\mathrm{first}(), e, W_3.\mathrm{last}())))).\mathrm{edges}()$ or $e \in W_3.\mathrm{edges}()$ or $e \in (W_1.\mathrm{cut}(W_1.\mathrm{findLastVertex}(G.\mathrm{walkOf}(W_3.\mathrm{first}(), e, W_3.\mathrm{last}())), \mathrm{len}\, W_1)).\mathrm{edges}()$. The theorem is a consequence of (37).

(39) Let us consider a graph G, walks W_1, W_3 of G, and an object e. Suppose e joins $W_3.\mathrm{first}()$ and $W_3.\mathrm{last}()$ in G and $e \notin W_3.\mathrm{edges}()$ and $G.\mathrm{walkOf}(W_3.\mathrm{first}(), e, W_3.\mathrm{last}())$ is an odd substring of W_1 not starting before 0 and for every even natural numbers n, m such that $n, m \in \mathrm{dom}\, W_1$ and $W_1(n) = e$ and $W_1(m) = e$ holds $n = m$.
Then $e \notin (W_1.\mathrm{replaceEdgeWith}(e, W_3)).\mathrm{edges}()$.
Proof: Set $W_2 = G.\mathrm{walkOf}(W_3.\mathrm{first}(), e, W_3.\mathrm{last}())$. W_2 is an odd substring of W_1 not starting before 1. Define $\mathcal{P}[\text{natural number}] \equiv \$_1$ is odd and $1 \leqslant \$_1 \leqslant \mathrm{len}\, W_1$ and $\mathrm{mid}(W_1, \$_1, \$_1 -' 1 + \mathrm{len}\, W_2) = W_2$. Consider i being a natural number such that $\mathcal{P}[i]$ and for every natural number n such that $\mathcal{P}[n]$ holds $i \leqslant n$. Set $j = i -' 1 + \mathrm{len}\, W_2$. W_2 is a substring of W_1. $1 \leqslant j \leqslant \mathrm{len}\, W_1$ and $i \leqslant j$. Set $n_1 = i + 1$. Reconsider $k = i - 1$ as an even natural number. For every natural number n such that $1 \leqslant n \leqslant \mathrm{len}\, W_2$ holds $W_1(k + n) = W_2(n)$. For every even natural number l such that for every natural number n such that $1 \leqslant n \leqslant \mathrm{len}\, W_2$ holds $W_1(l + n) = W_2(n)$ holds $k \leqslant l$. $i \leqslant \mathrm{len}\, W_1$ and there exists an even natural number k such that $i = k + 1$ and for every natural number n

such that $1 \leqslant n \leqslant \operatorname{len} W_2$ holds $W_1(k+n) = W_2(n)$ and for every even natural number l such that for every natural number n such that $1 \leqslant n \leqslant \operatorname{len} W_2$ holds $W_1(l+n) = W_2(n)$ holds $k \leqslant l$. W_1.findFirstVertex$(W_2) < n_1$. $n_1 \in \operatorname{dom} W_1$. $e \notin (W_1.\mathrm{cut}(1, W_1.\mathrm{findFirstVertex}(W_2)))$.edges(). $e \notin (W_1.\mathrm{cut}(W_1.\mathrm{findLastVertex}(W_2), \operatorname{len} W_1))$.edges() by [1, (4)], [6, (99)]. □

(40) Let us consider a graph G, a trail T_1 of G, a walk W_3 of G, and an object e. Suppose e joins W_3.first() and W_3.last() in G and $e \notin W_3$.edges() and $G.\mathrm{walkOf}(W_3.\mathrm{first}(), e, W_3.\mathrm{last}())$ is an odd substring of T_1 not starting before 0. Then $e \notin (T_1.\mathrm{replaceEdgeWith}(e, W_3))$.edges().

PROOF: For every even natural numbers n, m such that n, $m \in \operatorname{dom} T_1$ and $T_1(n) = e$ and $T_1(m) = e$ holds $n = m$. □

(41) Let us consider a graph G, and walks W_1, W_2 of G. Suppose W_1.first() = W_2.first() and W_1.last() = W_2.last(). Then $W_1.\mathrm{replaceWith}(W_1, W_2) = W_2$. The theorem is a consequence of (11), (32), and (31).

(42) Let us consider a graph G, walks W_1, W_3 of G, and an object e. Suppose e joins W_3.first() and W_3.last() in G and $G.\mathrm{walkOf}(W_3.\mathrm{first}(), e, W_3.\mathrm{last}())$ is an odd substring of W_1 not starting before 0. Then there exists a walk W_2 of G such that $W_1.\mathrm{replaceEdgeWith}(e, W_3) = W_1.\mathrm{replaceWith}(W_2, W_3)$.

(43) Let us consider a graph G, walks W_1, W_2 of G, and an object e. Suppose W_2 is an odd substring of W_1 not starting before 0 and e joins W_2.first() and W_2.last() in G. Then there exists a walk W_3 of G such that $W_1.\mathrm{replaceWithEdge}(W_2, e) = W_1.\mathrm{replaceWith}(W_2, W_3)$.

(44) Let us consider a graph G, walks W_1, W_3 of G, and an object e. Then

(i) W_1.first() = $(W_1.\mathrm{replaceEdgeWith}(e, W_3))$.first(), and

(ii) W_1.last() = $(W_1.\mathrm{replaceEdgeWith}(e, W_3))$.last().

The theorem is a consequence of (42) and (35).

(45) Let us consider a graph G, walks W_1, W_2 of G, and an object e. Then

(i) W_1.first() = $(W_1.\mathrm{replaceWithEdge}(W_2, e))$.first(), and

(ii) W_1.last() = $(W_1.\mathrm{replaceWithEdge}(W_2, e))$.last().

The theorem is a consequence of (43) and (35).

(46) Let us consider a graph G, walks W_1, W_2, W_3 of G, and objects u, v. Then W_1 is walk from u to v if and only if $W_1.\mathrm{replaceWith}(W_2, W_3)$ is walk from u to v. The theorem is a consequence of (35).

(47) Let us consider a graph G, walks W_1, W_3 of G, and objects e, u, v. Then W_1 is walk from u to v if and only if $W_1.\mathrm{replaceEdgeWith}(e, W_3)$ is walk from u to v. The theorem is a consequence of (42) and (46).

(48) Let us consider a graph G, walks W_1, W_2 of G, and objects e, u, v. Then W_1 is walk from u to v if and only if W_1.replaceWithEdge(W_2, e) is walk from u to v. The theorem is a consequence of (43) and (46).

(49) Let us consider a graph G, and vertices v_1, v_2 of G. Suppose v_1 is isolated and $v_1 \neq v_2$. Then $v_2 \notin G$.reachableFrom(v_1).

(50) Let us consider a graph G, and vertices v_1, v_2 of G. If $v_1 \in G$.reachableFrom(v_2), then $v_2 \in G$.reachableFrom(v_1).

(51) Let us consider a graph G, and a vertex v of G. If v is isolated, then $\{v\} = G$.reachableFrom(v).

PROOF: For every object x, $x \in \{v\}$ iff $x \in G$.reachableFrom(v) by [7, (9)], (49). □

(52) Let us consider a graph G, a vertex v of G, and a subgraph C of G induced by $\{v\}$. If v is isolated, then C is a component of G. The theorem is a consequence of (51).

(53) Let us consider a non trivial graph G_1, a vertex v of G_1, and a subgraph G_2 of G_1 with vertex v removed. Suppose v is isolated. Then

(i) G_1.componentSet() $= G_2$.componentSet() $\cup \{\{v\}\}$, and

(ii) G_1.numComponents() $= G_2$.numComponents() $+ 1$.

PROOF: For every object V, $V \in G_1$.componentSet() iff $V \in G_2$ .componentSet() $\cup \{\{v\}\}$. $\{v\} \notin G_2$.componentSet() by [8, (47)]. □

Let G be a graph. Let us observe that every vertex of G which is isolated is also non cut-vertex.

Now we state the propositions:

(54) Let us consider a graph G_1, a subgraph G_2 of G_1, a walk W_1 of G_1, and a walk W_2 of G_2. If $W_1 = W_2$, then W_1 is cycle-like iff W_2 is cycle-like.

(55) Let us consider a connected graph G_1, and a component G_2 of G_1. Then $G_1 \approx G_2$.

Observe that every graph which is complete is also connected and there exists a graph which is non non-directed-multi, non non-multi, non loopless, non directed-simple, non simple, non acyclic, and non finite.

From now on G denotes a graph.

Let us consider G. The functor G.endVertices() yielding a subset of the vertices of G is defined by

(Def. 8) for every object v, $v \in it$ iff there exists a vertex w of G such that $v = w$ and w is endvertex.

Now we state the proposition:

(56) Let us consider a vertex v of G. Then $v \in G$.endVertices() if and only if v is endvertex.

3. Supergraphs

Let us consider G.

A supergraph of G is a graph defined by

(Def. 9) the vertices of $G \subseteq$ the vertices of *it* and the edges of $G \subseteq$ the edges of *it* and for every set e such that $e \in$ the edges of G holds (the source of G)(e) = (the source of *it*)(e) and (the target of G)(e) = (the target of *it*)(e).

Let us consider graphs G_1, G_2. Now we state the propositions:

(57) G_2 is a subgraph of G_1 if and only if G_1 is a supergraph of G_2.

(58) G_2 is subgraph of G_1 and supergraph of G_1 if and only if $G_1 \approx G_2$. The theorem is a consequence of (57).

(59) G_1 is a supergraph of G_2 and G_2 is a supergraph of G_1 if and only if $G_1 \approx G_2$. The theorem is a consequence of (57).

(60) G_1 is a supergraph of G_2 if and only if $G_2 \subseteq G_1$. The theorem is a consequence of (57).

(61) G is a supergraph of G.

(62) Let us consider a graph G_3, and a supergraph G_2 of G_3. Then every supergraph of G_2 is a supergraph of G_3. The theorem is a consequence of (57).

In the sequel G_2 denotes a graph and G_1 denotes a supergraph of G_2.

(63) Let us consider graphs G_1, G_2. Suppose the vertices of $G_2 \subseteq$ the vertices of G_1 and the source of $G_2 \subseteq$ the source of G_1 and the target of $G_2 \subseteq$ the target of G_1. Then G_1 is a supergraph of G_2.

Let us consider G_2 and G_1. Now we state the propositions:

(64) (i) the source of $G_2 \subseteq$ the source of G_1, and

(ii) the target of $G_2 \subseteq$ the target of G_1.

(65) Suppose the vertices of G_2 = the vertices of G_1 and the edges of G_2 = the edges of G_1. Then $G_1 \approx G_2$. The theorem is a consequence of (64).

(66) Let us consider graphs G_1, G_2. Suppose the vertices of G_2 = the vertices of G_1 and the edges of G_2 = the edges of G_1 and the source of $G_2 \subseteq$ the source of G_1 and the target of $G_2 \subseteq$ the target of G_1. Then $G_1 \approx G_2$. The theorem is a consequence of (63) and (65).

(67) Let us consider a set x. Then

(i) if $x \in$ the vertices of G_2, then $x \in$ the vertices of G_1, and

(ii) if $x \in$ the edges of G_2, then $x \in$ the edges of G_1.

The theorem is a consequence of (57).

Let us consider G_2 and G_1. Now we state the propositions:

(68) Every vertex of G_2 is a vertex of G_1.

(69) (i) the source of G_2 = (the source of G_1)↾(the edges of G_2), and

(ii) the target of G_2 = (the target of G_1)↾(the edges of G_2).

The theorem is a consequence of (57).

(70) Let us consider sets x, y, and an object e. Then

(i) if e joins x and y in G_2, then e joins x and y in G_1, and

(ii) if e joins x to y in G_2, then e joins x to y in G_1, and

(iii) if e joins a vertex from x and a vertex from y in G_2, then e joins a vertex from x and a vertex from y in G_1, and

(iv) if e joins a vertex from x to a vertex from y in G_2, then e joins a vertex from x to a vertex from y in G_1.

The theorem is a consequence of (57).

Let us consider G_2, G_1, and objects e, v_1, v_2. Now we state the propositions:

(71) If e joins v_1 to v_2 in G_1, then e joins v_1 to v_2 in G_2 or $e \notin$ the edges of G_2.

(72) If e joins v_1 and v_2 in G_1, then e joins v_1 and v_2 in G_2 or $e \notin$ the edges of G_2. The theorem is a consequence of (71).

Let G be a finite graph. Observe that there exists a supergraph of G which is finite.

Now we state the propositions:

(73) (i) G_2.order() $\subseteq$ G_1.order(), and

(ii) G_2.size() $\subseteq$ G_1.size().

(74) Let us consider a finite graph G_2, and a finite supergraph G_1 of G_2. Then

(i) G_2.order() $\leqslant$ G_1.order(), and

(ii) G_2.size() $\leqslant$ G_1.size().

The theorem is a consequence of (57).

(75) Every walk of G_2 is a walk of G_1. The theorem is a consequence of (57).

(76) Let us consider a walk W_2 of G_2, and a walk W_1 of G_1. Suppose $W_1 = W_2$. Then

(i) W_1 is closed iff W_2 is closed, and

(ii) W_1 is directed iff W_2 is directed, and

(iii) W_1 is trivial iff W_2 is trivial, and

(iv) W_1 is trail-like iff W_2 is trail-like, and

(v) W_1 is path-like iff W_2 is path-like, and

(vi) W_1 is vertex-distinct iff W_2 is vertex-distinct, and

(vii) W_1 is cycle-like iff W_2 is cycle-like.

The theorem is a consequence of (57) and (54).

Let G be a non trivial graph. Note that every supergraph of G is non trivial.

Let G be a non non-directed-multi graph. Observe that every supergraph of G is non non-directed-multi.

Let G be a non non-multi graph. One can verify that every supergraph of G is non non-multi.

Let G be a non loopless graph. Let us note that every supergraph of G is non loopless.

Let G be a non directed-simple graph. Observe that every supergraph of G is non directed-simple.

Let G be a non simple graph. One can check that every supergraph of G is non simple.

Let G be a non acyclic graph. One can verify that every supergraph of G is non acyclic.

Every supergraph of a non finite graph G is non finite.

In the sequel V denotes a set. Let us consider G and V.

A supergraph of G extended by the vertices from V is a supergraph of G defined by

(Def. 10) the vertices of it = (the vertices of G)$\cup V$ and the edges of it = the edges of G and the source of it = the source of G and the target of it = the target of G.

Now we state the propositions:

(77) Let us consider supergraphs G_1, G_2 of G extended by the vertices from V. Then $G_1 \approx G_2$.

(78) Let us consider a supergraph G_1 of G_2 extended by the vertices from V. Then $G_1 \approx G_2$ if and only if $V \subseteq$ the vertices of G_2.

(79) Let us consider graphs G_1, G_2, and a set V. Suppose $G_1 \approx G_2$ and $V \subseteq$ the vertices of G_2. Then G_1 is a supergraph of G_2 extended by the vertices from V. The theorem is a consequence of (59).

(80) Let us consider a supergraph G_1 of G extended by the vertices from V. Suppose $G_1 \approx G_2$. Then G_2 is a supergraph of G extended by the vertices from V. The theorem is a consequence of (58) and (62).

(81) Let us consider a supergraph G_1 of G_2 extended by the vertices from V. Then G_1.edgesBetween(the vertices of G_2) = the edges of G_1.

PROOF: Set E_1 = the edges of G_1. Set V_2 = the vertices of G_2. For every object e, $e \in E_1$ iff $e \in G_1$.edgesInto(V_2) $\cap$ G_1.edgesOutOf(V_2). □

(82) Let us consider a graph G_3, sets V_1, V_2, and a supergraph G_2 of G_3 extended by the vertices from V_2. Then every supergraph of G_2 extended by the vertices from V_1 is a supergraph of G_3 extended by the vertices from $V_1 \cup V_2$. The theorem is a consequence of (62).

(83) Let us consider a graph G_3, sets V_1, V_2, and a supergraph G_1 of G_3 extended by the vertices from $V_1 \cup V_2$. Then there exists a supergraph G_2 of G_3 extended by the vertices from V_2 such that G_1 is a supergraph of G_2 extended by the vertices from V_1.

(84) Let us consider a supergraph G_1 of G_2 extended by the vertices from V. Then G_2 is a subgraph of G_1 induced by the vertices of G_2. The theorem is a consequence of (57) and (81).

(85) Let us consider a supergraph G_1 of G_2 extended by the vertices from V, and objects x, y, e. Then e joins x to y in G_1 if and only if e joins x to y in G_2.

(86) Let us consider a supergraph G_1 of G_2 extended by the vertices from V, and an object v. If $v \in V$, then v is a vertex of G_1.

(87) Let us consider a supergraph G_1 of G_2 extended by the vertices from V, and objects x, y, e. Then e joins x and y in G_1 if and only if e joins x and y in G_2. The theorem is a consequence of (85).

(88) Let us consider a supergraph G_1 of G_2 extended by the vertices from V, and a vertex v of G_1. Suppose $v \in V \setminus$ (the vertices of G_2). Then v is isolated and non cut-vertex.

Proof: v.edgesInOut() $= \emptyset$. □

(89) Let us consider a supergraph G_1 of G_2 extended by the vertices from V. Suppose $V \setminus$ (the vertices of G_2) $\neq \emptyset$. Then G_1 is non trivial, non connected, non tree-like, and non complete.

Proof: Consider v_1 being an object such that $v_1 \in V \setminus$ (the vertices of G_2). $\overline{\overline{\alpha}} \neq 1$, where α is the vertices of G_1. v_1 is isolated. □

Let G be a non-directed-multi graph and V be a set. Note that every supergraph of G extended by the vertices from V is non-directed-multi.

Let G be a non-multi graph. One can verify that every supergraph of G extended by the vertices from V is non-multi.

Let G be a loopless graph. Observe that every supergraph of G extended by the vertices from V is loopless.

Let G be a directed-simple graph. Let us note that every supergraph of G extended by the vertices from V is directed-simple.

Let G be a simple graph. Let us note that every supergraph of G extended by the vertices from V is simple.

Let us consider G_2, V, a supergraph G_1 of G_2 extended by the vertices from V, and a walk W of G_1. Now we state the propositions:

(90) (i) W.vertices() misses $V \setminus$ (the vertices of G_2), or

(ii) W is trivial.

The theorem is a consequence of (85).

(91) If W.vertices() misses $V \setminus$ (the vertices of G_2), then W is a walk of G_2. The theorem is a consequence of (57).

Let G be an acyclic graph and V be a set. Let us note that every supergraph of G extended by the vertices from V is acyclic.

(92) Let us consider a supergraph G_1 of G_2 extended by the vertices from V. Then G_2 is chordal if and only if G_1 is chordal.

PROOF: If G_2 is chordal, then G_1 is chordal. G_2 is a subgraph of G_1 induced by the vertices of G_2. □

Let G be a chordal graph and V be a set. Let us observe that every supergraph of G extended by the vertices from V is chordal.

From now on v denotes an object.

Let us consider G and v.

A supergraph of G extended by v is a supergraph of G extended by the vertices from $\{v\}$.

Let us consider G_2, v, and a supergraph G_1 of G_2 extended by v. Now we state the propositions:

(93) $G_1 \approx G_2$ if and only if $v \in$ the vertices of G_2.

(94) v is a vertex of G_1. The theorem is a consequence of (86).

Let us consider G. One can verify that every supergraph of G extended by the vertices of G is non trivial, non connected, and non complete and there exists a graph which is non trivial, non connected, and non complete.

Let G be a non connected graph and V be a set. Note that every supergraph of G extended by the vertices from V is non connected.

Now we state the propositions:

(95) Let us consider a supergraph G_1 of G_2 extended by the vertices from V. Then

(i) G_1.size() $= G_2$.size(), and

(ii) G_1.order() $= G_2$.order() $+ \overline{\overline{V \setminus \alpha}}$,

where α is the vertices of G_2.

(96) Let us consider a finite graph G_2, a finite set V, and a supergraph G_1 of G_2 extended by the vertices from V. Then G_1.order() $= G_2$.order() $+ \overline{\overline{V \setminus \alpha}}$, where α is the vertices of G_2.

(97) Let us consider a graph G_2, an object v, and a supergraph G_1 of G_2 extended by v. Suppose $v \notin$ the vertices of G_2. Then G_1.order() $= G_2$.order() $+ 1$. The theorem is a consequence of (95).

(98) Let us consider a finite graph G_2, an object v, and a supergraph G_1 of G_2 extended by v. Suppose $v \notin$ the vertices of G_2. Then G_1.order() $=$ G_2.order() $+ 1$. The theorem is a consequence of (96).

Let G be a finite graph and V be a finite set. Note that every supergraph of G extended by the vertices from V is finite.

Let v be an object. Note that every supergraph of G extended by v is finite.

Let G be a graph and V be a non finite set. Note that every supergraph of G extended by the vertices from V is non finite.

Let us consider G. Let v_1, e, v_2 be objects.

A supergraph of G extended by e between vertices v_1 and v_2 is a supergraph of G defined by

(Def. 11) (i) the vertices of *it* = the vertices of G and the edges of *it* = (the edges of G) $\cup \{e\}$ and the source of *it* = (the source of G)$+\cdot(e \dot{\longmapsto} v_1)$ and the target of *it* = (the target of G)$+\cdot(e \dot{\longmapsto} v_2)$, **if** v_1, $v_2 \in$ the vertices of G and $e \notin$ the edges of G,

(ii) *it* $\approx G$, **otherwise**.

Now we state the propositions:

(99) Let us consider objects v_1, e, v_2, and supergraphs G_1, G_2 of G extended by e between vertices v_1 and v_2. Then $G_1 \approx G_2$.

(100) Let us consider vertices v_1, v_2 of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2. Then $G_1 \approx G_2$ if and only if $e \in$ the edges of G_2.

(101) Let us consider objects v_1, e, v_2, and a supergraph G_1 of G extended by e between vertices v_1 and v_2. Suppose $G_1 \approx G_2$. Then G_2 is a supergraph of G extended by e between vertices v_1 and v_2. The theorem is a consequence of (58) and (62).

Let us consider G_2, vertices v_1, v_2 of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2. Now we state the propositions:

(102) The vertices of G_1 = the vertices of G_2.

(103) G_1.edgesBetween(the vertices of G_2) = the edges of G_1. The theorem is a consequence of (102).

(104) Every vertex of G_1 is a vertex of G_2.

(105) If $e \notin$ the edges of G_2, then e joins v_1 to v_2 in G_1.

Let us consider G_2, vertices v_1, v_2 of G_2, an object e, a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2, and objects e_1, w_1, w_2. Now we state the propositions:

(106) Suppose $e \notin$ the edges of G_2. Then if e_1 joins w_1 and w_2 in G_1 and $e_1 \notin$ the edges of G_2, then $e_1 = e$.

(107) Suppose $e \notin$ the edges of G_2. Then suppose e_1 joins w_1 and w_2 in G_1 and $e_1 \notin$ the edges of G_2. Then

(i) $w_1 = v_1$ and $w_2 = v_2$, or

(ii) $w_1 = v_2$ and $w_2 = v_1$.

The theorem is a consequence of (106) and (105).

(108) Let us consider vertices v_1, v_2 of G_2, a set e, and a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2. Suppose $e \notin$ the edges of G_2. Then G_2 is a subgraph of G_1 with edge e removed. The theorem is a consequence of (57).

(109) Let us consider vertices v_1, v_2 of G_2, an object e, a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2, and a walk W of G_1. Suppose if $e \in W$.edges(), then $e \in$ the edges of G_2. Then W is a walk of G_2. The theorem is a consequence of (57).

Let G be a trivial graph and v_1, e, v_2 be objects. Let us note that every supergraph of G extended by e between vertices v_1 and v_2 is trivial.

Let G be a connected graph. Let us note that every supergraph of G extended by e between vertices v_1 and v_2 is connected.

Let G be a complete graph. Note that every supergraph of G extended by e between vertices v_1 and v_2 is complete.

Now we state the propositions:

(110) Let us consider vertices v_1, v_2 of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2. Suppose $e \notin$ the edges of G_2. Then

(i) G_1.order() $= G_2$.order(), and

(ii) G_1.size() $= G_2$.size() $+ 1$.

(111) Let us consider a finite graph G_2, vertices v_1, v_2 of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2. Suppose $e \notin$ the edges of G_2. Then G_1.size() $= G_2$.size() $+ 1$.

Let G be a finite graph and v_1, e, v_2 be objects. Observe that every supergraph of G extended by e between vertices v_1 and v_2 is finite.

(112) Let us consider vertices v_1, v_2 of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2. If G_2 is loopless and $v_1 \neq v_2$, then G_1 is loopless. The theorem is a consequence of (105).

(113) Let us consider a vertex v of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v and v. Suppose G_2 is not loopless or $e \notin$ the edges of G_2. Then G_1 is not loopless. The theorem is a consequence of (105).

Let us consider G. Let v be a vertex of G. Let us note that every supergraph of G extended by the edges of G between vertices v and v is non loopless.

Let us consider G_2, vertices v_1, v_2 of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2. Now we state the propositions:

(114) If G_2 is non-directed-multi and there exists no object e_3 such that e_3 joins v_1 to v_2 in G_2, then G_1 is non-directed-multi. The theorem is a consequence of (71) and (105).

(115) Suppose $e \notin$ the edges of G_2 and there exists an object e_2 such that e_2 joins v_1 to v_2 in G_2. Then G_1 is not non-directed-multi. The theorem is a consequence of (105) and (70).

(116) If G_2 is non-multi and v_1 and v_2 are not adjacent, then G_1 is non-multi. The theorem is a consequence of (72) and (105).

(117) If $e \notin$ the edges of G_2 and v_1 and v_2 are adjacent, then G_1 is not non-multi.

Proof: There exist objects e_1, e_2, u_1, u_2 such that e_1 joins u_1 and u_2 in G_1 and e_2 joins u_1 and u_2 in G_1 and $e_1 \neq e_2$. □

(118) If G_2 is acyclic and $v_2 \notin G_2$.reachableFrom(v_1), then G_1 is acyclic. The theorem is a consequence of (57), (54), and (105).

(119) If $e \notin$ the edges of G_2 and $v_2 \in G_2$.reachableFrom(v_1), then G_1 is not acyclic. The theorem is a consequence of (75), (105), and (113).

(120) If G_2 is not connected and $v_2 \in G_2$.reachableFrom(v_1), then G_1 is not connected. The theorem is a consequence of (68), (109), (27), (105), (75), (47), and (40).

(121) Suppose $e \notin$ the edges of G_2 and for every vertices v_3, v_4 of G_2 such that v_3 and v_4 are not adjacent holds $v_3 = v_4$ or $v_1 = v_3$ and $v_2 = v_4$ or $v_1 = v_4$ and $v_2 = v_3$. Then G_1 is complete.

Proof: For every vertices u_1, u_2 of G_1 such that $u_1 \neq u_2$ holds u_1 and u_2 are adjacent. □

(122) If G_2 is not complete and v_1 and v_2 are adjacent, then G_1 is not complete. The theorem is a consequence of (68), (72), and (105).

Let us consider G. Let v_1, e, v_2 be objects.

A supergraph of G extended by v_1, v_2 and e between them is a supergraph of G defined by

(Def. 12) (i) the vertices of *it* = (the vertices of G) $\cup$ $\{v_2\}$ and the edges of

it = (the edges of G) $\cup \{e\}$ and the source of it = (the source of G)$+\cdot$($e \dot{\longmapsto} v_1$) and the target of it = (the target of G)$+\cdot$($e \dot{\longmapsto} v_2$), **if** $v_1 \in$ the vertices of G and $v_2 \notin$ the vertices of G and $e \notin$ the edges of G,

(ii) the vertices of it = (the vertices of G) $\cup \{v_1\}$ and the edges of it = (the edges of G) $\cup \{e\}$ and the source of it = (the source of G)$+\cdot$($e \dot{\longmapsto} v_1$) and the target of it = (the target of G)$+\cdot$($e \dot{\longmapsto} v_2$), **if** $v_1 \notin$ the vertices of G and $v_2 \in$ the vertices of G and $e \notin$ the edges of G,

(iii) $it \approx G$, **otherwise**.

Let v_1 be a vertex of G and e, v_2 be objects.

One can check that a supergraph of G extended by v_1, v_2 and e between them can equivalently be formulated as follows:

(Def. 13) (i) the vertices of it = (the vertices of G) $\cup \{v_2\}$ and the edges of it = (the edges of G) $\cup \{e\}$ and the source of it = (the source of G)$+\cdot$($e \dot{\longmapsto} v_1$) and the target of it = (the target of G)$+\cdot$($e \dot{\longmapsto} v_2$), **if** $v_2 \notin$ the vertices of G and $e \notin$ the edges of G,

(ii) $it \approx G$, **otherwise**.

Let v_1, e be objects and v_2 be a vertex of G.

Let us note that a supergraph of G extended by v_1, v_2 and e between them can equivalently be formulated as follows:

(Def. 14) (i) the vertices of it = (the vertices of G) $\cup \{v_1\}$ and the edges of it = (the edges of G) $\cup \{e\}$ and the source of it = (the source of G)$+\cdot$($e \dot{\longmapsto} v_1$) and the target of it = (the target of G)$+\cdot$($e \dot{\longmapsto} v_2$), **if** $v_1 \notin$ the vertices of G and $e \notin$ the edges of G,

(ii) $it \approx G$, **otherwise**.

Now we state the propositions:

(123) Let us consider objects v_1, e, v_2, and supergraphs G_1, G_2 of G extended by v_1, v_2 and e between them. Then $G_1 \approx G_2$.

(124) Let us consider objects v_1, e, v_2, and a supergraph G_1 of G extended by v_1, v_2 and e between them. Suppose $G_1 \approx G_2$. Then G_2 is a supergraph of G extended by v_1, v_2 and e between them. The theorem is a consequence of (58) and (62).

(125) Let us consider a vertex v_1 of G_2, objects e, v_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $e \notin$ the edges of G_2 and $v_2 \notin$ the vertices of G_2. Then there exists a supergraph G_3 of G_2 extended by v_2 such that G_1 is a supergraph of G_3 extended by e between vertices v_1 and v_2. The theorem is a consequence of (94).

(126) Let us consider objects v_1, e, a vertex v_2 of G_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $e \notin$ the edges of G_2 and $v_1 \notin$ the vertices of G_2. Then there exists a supergraph G_3 of G_2 extended by v_1 such that G_1 is a supergraph of G_3 extended by e between vertices v_1 and v_2. The theorem is a consequence of (94).

(127) Let us consider a graph G_3, a vertex v_1 of G_3, objects e, v_2, a supergraph G_2 of G_3 extended by v_2, and a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2. Suppose $e \notin$ the edges of G_3 and $v_2 \notin$ the vertices of G_3. Then G_1 is a supergraph of G_3 extended by v_1, v_2 and e between them. The theorem is a consequence of (62), (68), and (94).

(128) Let us consider a graph G_3, objects v_1, e, a vertex v_2 of G_3, a supergraph G_2 of G_3 extended by v_1, and a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2. Suppose $e \notin$ the edges of G_3 and $v_1 \notin$ the vertices of G_3. Then G_1 is a supergraph of G_3 extended by v_1, v_2 and e between them. The theorem is a consequence of (62), (68), and (94).

(129) Let us consider a vertex v_1 of G_2, objects e, v_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $v_2 \notin$ the vertices of G_2 and $e \notin$ the edges of G_2. Then v_2 is a vertex of G_1.

(130) Let us consider objects v_1, e, a vertex v_2 of G_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $v_1 \notin$ the vertices of G_2 and $e \notin$ the edges of G_2. Then v_1 is a vertex of G_1.

(131) Let us consider a vertex v_1 of G_2, objects e, v_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $v_2 \notin$ the vertices of G_2 and $e \notin$ the edges of G_2. Then

(i) e joins v_1 to v_2 in G_1, and

(ii) e joins v_1 and v_2 in G_1.

(132) Let us consider objects v_1, e, a vertex v_2 of G_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $v_1 \notin$ the vertices of G_2 and $e \notin$ the edges of G_2. Then

(i) e joins v_1 to v_2 in G_1, and

(ii) e joins v_1 and v_2 in G_1.

(133) Let us consider a vertex v_1 of G_2, objects e, v_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $v_2 \notin$ the vertices of G_2 and $e \notin$ the edges of G_2. Let us consider objects e_1, w. If $w \neq v_1$ or $e_1 \neq e$, then e_1 does not join w and v_2 in G_1. The theorem is a consequence of (72) and (131).

(134) Let us consider objects v_1, e, a vertex v_2 of G_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $v_1 \notin$ the vertices of

G_2 and $e \notin$ the edges of G_2. Let us consider objects e_1, w. If $w \neq v_2$ or $e_1 \neq e$, then e_1 does not join v_1 and w in G_1. The theorem is a consequence of (72) and (132).

Let us consider G_2, objects v_1, e, v_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Now we state the propositions:

(135) G_1.edgesBetween(the vertices of G_2) = the edges of G_2. The theorem is a consequence of (131), (70), and (132).

(136) G_2 is a subgraph of G_1 induced by the vertices of G_2. The theorem is a consequence of (57), (135), (15), and (16).

(137) Let us consider a vertex v_1 of G_2, an object e, a set v_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $e \notin$ the edges of G_2 and $v_2 \notin$ the vertices of G_2. Then G_2 is a subgraph of G_1 with vertex v_2 removed. The theorem is a consequence of (136).

(138) Let us consider a set v_1, an object e, a vertex v_2 of G_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $e \notin$ the edges of G_2 and $v_1 \notin$ the vertices of G_2. Then G_2 is a subgraph of G_1 with vertex v_1 removed. The theorem is a consequence of (136).

(139) Let us consider a non trivial graph G, a vertex v_1 of G, objects e, v_2, a supergraph G_1 of G extended by v_1, v_2 and e between them, a subgraph G_2 of G_1 with vertex v_1 removed, and a subgraph G_3 of G with vertex v_1 removed. Suppose $e \notin$ the edges of G and $v_2 \notin$ the vertices of G. Then G_2 is a supergraph of G_3 extended by v_2.

PROOF: v_1 is a vertex of G_1 and $v_1 \neq v_2$. For every object e_1, $e_1 \in$ G_1.edgesBetween((the vertices of G_1)$\setminus\{v_1\}$) iff $e_1 \in G$.edgesBetween((the vertices of G)$\setminus\{v_1\}$). For every object e_1 such that $e_1 \in$ dom(the source of G_2) holds (the source of G_2)(e_1) = (the source of G_3)(e_1). For every object e_1 such that $e_1 \in$ dom(the target of G_2) holds (the target of G_2)(e_1) = (the target of G_3)(e_1). □

(140) Let us consider a non trivial graph G, objects v_1, e, a vertex v_2 of G, a supergraph G_1 of G extended by v_1, v_2 and e between them, a subgraph G_2 of G_1 with vertex v_2 removed, and a subgraph G_3 of G with vertex v_2 removed. Suppose $e \notin$ the edges of G and $v_1 \notin$ the vertices of G. Then G_2 is a supergraph of G_3 extended by v_1.

PROOF: v_2 is a vertex of G_1 and $v_1 \neq v_2$. For every object e_1, $e_1 \in$ G_1.edgesBetween((the vertices of G_1)$\setminus\{v_2\}$) iff $e_1 \in G$.edgesBetween((the vertices of G)$\setminus\{v_2\}$). For every object e_1 such that $e_1 \in$ dom(the source of G_2) holds (the source of G_2)(e_1) = (the source of G_3)(e_1). For every object e_1 such that $e_1 \in$ dom(the target of G_2) holds (the target of G_2)(e_1) = (the target of G_3)(e_1). □

(141) Let us consider a vertex v_1 of G_2, objects e, v_2, a supergraph G_1 of G_2 extended by v_1, v_2 and e between them, and a vertex w of G_1. Suppose $e \notin$ the edges of G_2 and $v_2 \notin$ the vertices of G_2 and $w = v_2$. Then w is endvertex.

PROOF: There exists an object e_1 such that w.edgesInOut() $= \{e_1\}$ and e_1 does not join w and w in G_1. $\square$

(142) Let us consider objects v_1, e, a vertex v_2 of G_2, a supergraph G_1 of G_2 extended by v_1, v_2 and e between them, and a vertex w of G_1. Suppose $e \notin$ the edges of G_2 and $v_1 \notin$ the vertices of G_2 and $w = v_1$. Then w is endvertex.

PROOF: There exists an object e_1 such that w.edgesInOut() $= \{e_1\}$ and e_1 does not join w and w in G_1. $\square$

(143) Let us consider a vertex v_1 of G_2, objects e, v_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $v_2 \notin$ the vertices of G_2 and $e \notin$ the edges of G_2. Then G_1 is not trivial. The theorem is a consequence of (125) and (89).

(144) Let us consider objects v_1, e, a vertex v_2 of G_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $v_1 \notin$ the vertices of G_2 and $e \notin$ the edges of G_2. Then G_1 is not trivial. The theorem is a consequence of (126) and (89).

Let G be a graph and v be a vertex of G. Let us note that every supergraph of G extended by v, the vertices of G and the edges of G between them is non trivial and every supergraph of G extended by the vertices of G, v and the edges of G between them is non trivial.

Let G be a trivial graph. Note that every supergraph of G extended by v, the vertices of G and the edges of G between them is complete and every supergraph of G extended by the vertices of G, v and the edges of G between them is complete.

Let G be a loopless graph and v_1, e, v_2 be objects. One can verify that every supergraph of G extended by v_1, v_2 and e between them is loopless.

Let G be a non-directed-multi graph. One can check that every supergraph of G extended by v_1, v_2 and e between them is non-directed-multi.

Let G be a non-multi graph. One can check that every supergraph of G extended by v_1, v_2 and e between them is non-multi.

Let G be a directed-simple graph. One can check that every supergraph of G extended by v_1, v_2 and e between them is directed-simple.

Let G be a simple graph. One can check that every supergraph of G extended by v_1, v_2 and e between them is simple.

Now we state the propositions:

(145) Let us consider a vertex v_1 of G_2, objects e, v_2, a supergraph G_1 of G_2 extended by v_1, v_2 and e between them, and a walk W of G_1. Suppose $e \notin$ the edges of G_2 and $v_2 \notin$ the vertices of G_2 and ($e \notin W$.edges() and W is not trivial or $v_2 \notin W$.vertices()). Then W is a walk of G_2. The theorem is a consequence of (125), (68), (94), (108), (90), (91), (137), (129), and (143).

(146) Let us consider objects v_1, e, a vertex v_2 of G_2, a supergraph G_1 of G_2 extended by v_1, v_2 and e between them, and a walk W of G_1. Suppose $e \notin$ the edges of G_2 and $v_1 \notin$ the vertices of G_2 and ($e \notin W$.edges() and W is not trivial or $v_1 \notin W$.vertices()). Then W is a walk of G_2. The theorem is a consequence of (126), (68), (94), (108), (90), (91), (138), (130), and (144).

(147) Let us consider objects v_1, e, v_2, a supergraph G_1 of G_2 extended by v_1, v_2 and e between them, and a trail T of G_1. Suppose $e \notin$ the edges of G_2 and T.first(), T.last() $\in$ the vertices of G_2. Then $e \notin T$.edges(). The theorem is a consequence of (129), (141), (145), (130), (142), and (146).

Let G be a connected graph and v_1, e, v_2 be objects. Let us observe that every supergraph of G extended by v_1, v_2 and e between them is connected.

Let G be a non connected graph. One can check that every supergraph of G extended by v_1, v_2 and e between them is non connected.

Let G be an acyclic graph. Note that every supergraph of G extended by v_1, v_2 and e between them is acyclic.

Let G be a tree-like graph. One can verify that every supergraph of G extended by v_1, v_2 and e between them is tree-like.

Now we state the propositions:

(148) Let us consider a vertex v_1 of G_2, objects e, v_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $e \notin$ the edges of G_2 and $v_2 \notin$ the vertices of G_2 and G_2 is not trivial. Then G_1 is not complete.
Proof: There exist vertices u, v of G_1 such that $u \neq v$ and u and v are not adjacent. □

(149) Let us consider objects v_1, e, a vertex v_2 of G_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $e \notin$ the edges of G_2 and $v_1 \notin$ the vertices of G_2 and G_2 is not trivial. Then G_1 is not complete.
Proof: There exist vertices u, v of G_1 such that $u \neq v$ and u and v are not adjacent. □

Let G be a non complete graph and v_1, e, v_2 be objects. Observe that every supergraph of G extended by v_1, v_2 and e between them is non complete.

Let v be a vertex of G. Observe that every supergraph of G extended by v, the vertices of G and the edges of G between them is non complete and every supergraph of G extended by the vertices of G, v and the edges of G between them is non complete.

Now we state the propositions:

(150) Let us consider a vertex v_1 of G_2, objects e, v_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $e \notin$ the edges of G_2 and $v_2 \notin$ the vertices of G_2. Then

(i) G_1.order() $= G_2$.order() $+ 1$, and

(ii) G_1.size() $= G_2$.size() $+ 1$.

(151) Let us consider objects v_1, e, a vertex v_2 of G_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $e \notin$ the edges of G_2 and $v_1 \notin$ the vertices of G_2. Then

(i) G_1.order() $= G_2$.order() $+ 1$, and

(ii) G_1.size() $= G_2$.size() $+ 1$.

(152) Let us consider a finite graph G_2, a vertex v_1 of G_2, objects e, v_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $e \notin$ the edges of G_2 and $v_2 \notin$ the vertices of G_2. Then

(i) G_1.order() $= G_2$.order() $+ 1$, and

(ii) G_1.size() $= G_2$.size() $+ 1$.

(153) Let us consider a finite graph G_2, objects v_1, e, a vertex v_2 of G_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $e \notin$ the edges of G_2 and $v_1 \notin$ the vertices of G_2. Then

(i) G_1.order() $= G_2$.order() $+ 1$, and

(ii) G_1.size() $= G_2$.size() $+ 1$.

Let G be a finite graph and v_1, e, v_2 be objects. One can verify that every supergraph of G extended by v_1, v_2 and e between them is finite.

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