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# On Roots of Polynomials and Algebraically Closed Fields

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**Summary.** In this article we further extend the algebraic theory of polynomial rings in Mizar [1, 2, 3]. We deal with roots and multiple roots of polynomials and show that both the real numbers and finite domains are not algebraically closed [5, 7]. We also prove the identity theorem for polynomials and that the number of multiple roots is bounded by the polynomial's degree [4, 6].

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## 1. Preliminaries

From now on $n$ denotes a natural number.

Note that there exists a natural number which is non trivial and non prime.

Now we state the proposition:

(1) Let us consider an even natural number $n$, and an element $x$ of $\mathbb{R}_{\rm F}$. Then $x^n \geqslant 0_{\mathbb{R}_{\rm F}}$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv x^{2\cdot\$_1} \geqslant 0_{\mathbb{R}_{\rm F}}$. For every element $x$ of $\mathbb{R}_{\rm F}$, $x^2 \geqslant 0_{\mathbb{R}_{\rm F}}$. For every natural number $k$, $\mathcal{P}[k]$. $\square$

Let us consider a ring $R$ and an element $a$ of $R$. Now we state the propositions:

(2) $2 \star a = a + a$.

(3) $a^2 = a \cdot a$.




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Let $F$ be a field and $a$ be an element of $F$. Note that $\frac{a}{1_F}$ reduces to $a$.

One can check that $\mathbb{Z}/2$ is non trivial and almost left invertible.

Let $n$ be a non trivial, non prime natural number. Note that $\mathbb{Z}/n$ is non integral domain-like and $\mathbb{Z}/6$ is non degenerated.

## 2. Some More Properties of Polynomials

Let $R$ be a non degenerated ring. Observe that every non zero polynomial over $R$ is non-zero and every polynomial over $R$ which is monic is also non zero.

Let $p$ be a non zero polynomial over $R$. One can check that $\deg p$ is natural.

Let $R$ be a ring, $p$ be a zero polynomial over $R$, and $q$ be a polynomial over $R$. Let us observe that $p * q$ is zero and $q * p$ is zero.

Let us observe that $p + q$ reduces to $q$ and $q + p$ reduces to $q$.

Let $p$ be a polynomial over $R$. One can check that $p * \mathbf{0}.R$ reduces to $\mathbf{0}.R$ and $p * \mathbf{1}.R$ reduces to $p$ and $\mathbf{0}.R * p$ reduces to $\mathbf{0}.R$ and $\mathbf{1}.R * p$ reduces to $p$.

One can check that $1_R \cdot p$ reduces to $p$.

Now we state the propositions:

(4) Let us consider an integral domain $R$, a polynomial $p$ over $R$, and a non zero element $a$ of $R$. Then $\deg(a \cdot p) = \deg p$.

(5) Let us consider an integral domain $R$, a polynomial $p$ over $R$, and an element $a$ of $R$. Then $\mathrm{LC}(a \cdot p) = a \cdot \mathrm{LC}\, p$.

(6) Let us consider an integral domain $R$, and an element $a$ of $R$. Then $\mathrm{LC}(a{\restriction}R) = a$. The theorem is a consequence of (5).

(7) Let us consider an integral domain $R$, a polynomial $p$ over $R$, and elements $v$, $x$ of $R$. Then $\mathrm{eval}(v \cdot p, x) = v \cdot \mathrm{eval}(p, x)$. The theorem is a consequence of (4).

(8) Let us consider a ring $R$, and elements $a$, $b$ of $R$. Then $\mathrm{eval}(a{\restriction}R, b) = a$.

Let $R$ be an integral domain and $p$, $q$ be monic polynomials over $R$. Let us note that $p * q$ is monic.

Let $a$ be an element of $R$ and $k$ be a natural number. One can check that $(\mathrm{rpoly}(1, a))^k$ is non zero and monic.

Now we state the propositions:

(9) Let us consider a non degenerated ring $R$, an element $a$ of $R$, and a non zero element $k$ of $\mathbb{N}$. Then $\mathrm{LC}\,\mathrm{rpoly}(k, a) = 1_R$.

(10) Let us consider a non degenerated, well unital, non empty double loop structure $R$, and an element $a$ of $R$. Then $\langle -a, 1_R \rangle = \mathrm{rpoly}(1, a)$.

(11) Let us consider an integral domain $R$, a polynomial $p$ over $R$, and an element $x$ of $R$. Then $\mathrm{eval}(p, x) = 0_R$ if and only if $\mathrm{rpoly}(1, x) \mid p$.

(12) Let us consider an integral domain $F$, polynomials $p$, $q$ over $F$, and an element $a$ of $F$. Suppose $\mathrm{rpoly}(1, a) \mid p * q$. Then

(i) $\mathrm{rpoly}(1, a) \mid p$, or

(ii) $\mathrm{rpoly}(1, a) \mid q$.

The theorem is a consequence of (11).

(13) Let us consider an integral domain $R$, a polynomial $p$ over $R$, and a non zero polynomial $q$ over $R$. If $p \mid q$, then $\deg p \leqslant \deg q$.

(14) Let us consider a non degenerated commutative ring $R$, a polynomial $q$ over $R$, a non zero polynomial $p$ over $R$, and a non zero element $b$ of $R$. If $q \mid p$, then $q \mid b \cdot p$.

(15) Let us consider a field $F$, a polynomial $q$ over $F$, a non zero polynomial $p$ over $F$, and a non zero element $b$ of $F$. Then $q \mid p$ if and only if $q \mid b \cdot p$. The theorem is a consequence of (14).

Let us consider an integral domain $R$, a non zero polynomial $p$ over $R$, an element $a$ of $R$, and a non zero element $b$ of $R$. Now we state the propositions:

(16) $\mathrm{rpoly}(1, a) \mid p$ if and only if $\mathrm{rpoly}(1, a) \mid b \cdot p$. The theorem is a consequence of (11), (7), and (14).

(17) $(\mathrm{rpoly}(1, a))^n \mid p$ if and only if $(\mathrm{rpoly}(1, a))^n \mid b \cdot p$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $(\mathrm{rpoly}(1, a))^{\$_1} \mid b \cdot p$, then $(\mathrm{rpoly}(1, a))^{\$_1} \mid p$. For every natural number $k$, $\mathcal{P}[k]$. □

Let $R$ be an integral domain, $p$ be a non zero polynomial over $R$, and $b$ be a non zero element of $R$. Let us note that $b \cdot p$ is non zero.

## 3. On Roots of Polynomials

Let $R$ be a non degenerated ring. One can check that $\mathbf{1}.R$ and has not roots.

Let $a$ be a non zero element of $R$. One can verify that $a \restriction R$ and has not roots and every polynomial over $R$ which is non zero and has roots is also non constant and every polynomial over $R$ which and has not roots is also non zero.

Let $a$ be an element of $R$. One can check that $\mathrm{rpoly}(1, a)$ is non zero and has roots and there exists a polynomial over $R$ which is non zero and has not roots and there exists a polynomial over $R$ which is non zero and has roots.

Let $R$ be an integral domain, $p$ be a polynomial over $R$ with non roots, and $a$ be a non zero element of $R$. Let us note that $a \cdot p$ and has not roots.

Let $p$ be a polynomial over $R$ with roots and $a$ be an element of $R$. Note that $a \cdot p$ has roots.

Let $R$ be a non degenerated commutative ring and $q$ be a polynomial over $R$. One can verify that $p * q$ has roots.

Let $R$ be an integral domain and $p$, $q$ be polynomials over $R$ with non roots. One can check that $p * q$ and has not roots.

Let $R$ be a non degenerated commutative ring, $a$ be an element of $R$, and $k$ be a non zero element of $\mathbb{N}$. Let us note that $\mathrm{rpoly}(k, a)$ is non constant and monic and has roots.

Let $R$ be a non degenerated ring. Let us observe that there exists a polynomial over $R$ which is non constant and monic.

Let $R$ be an integral domain, $a$ be an element of $R$, $k$ be a non zero natural number, and $n$ be a non zero element of $\mathbb{N}$. Note that $(\mathrm{rpoly}(n, a))^k$ is non constant and monic and has roots.

Let $R$ be a ring and $p$ be a polynomial over $R$ with roots. Note that $\mathrm{Roots}(p)$ is non empty.

Let $R$ be a non degenerated ring and $p$ be a polynomial over $R$ with non roots. Let us observe that $\mathrm{Roots}(p)$ is empty.

Let $R$ be an integral domain. One can check that there exists a polynomial over $R$ which is monic and has roots and there exists a polynomial over $R$ which is monic and has not roots.

Now we state the propositions:

(18) Let us consider a non degenerated ring $R$, and an element $a$ of $R$. Then $\mathrm{Roots}(\mathrm{rpoly}(1, a)) = \{a\}$.

(19) Let us consider an integral domain $F$, a polynomial $p$ over $F$, and a non zero element $b$ of $F$. Then $\mathrm{Roots}(b \cdot p) = \mathrm{Roots}(p)$. The theorem is a consequence of (7).

(20) There exist polynomials $p$, $q$ over $\mathbb{Z}/6$ such that $\mathrm{Roots}(p*q) \not\subseteq \mathrm{Roots}(p) \cup \mathrm{Roots}(q)$.

(21) Let us consider an integral domain $R$, and elements $a$, $b$ of $R$. Then $\mathrm{rpoly}(1, a) \mid \mathrm{rpoly}(1, b)$ if and only if $a = b$. The theorem is a consequence of (18).

(22) Let us consider an integral domain $R$, and a non zero polynomial $p$ over $R$. Then $\overline{\overline{\mathrm{Roots}(p)}} \leqslant \deg p$.

## 4. More about Bags

Let $X$ be a non empty set and $B$ be a bag of $X$. We introduce the notation $\overline{\overline{B}}$ as a synonym of $\sum B$.

Observe that there exists a bag of $X$ which is zero and there exists a bag of $X$ which is non zero.

Let $b_1$ be a bag of $X$ and $b_2$ be a bag of $X$. One can check that $b_1 + b_2$ is $X$-defined and $b_1 + b_2$ is total.

Let us consider a non empty set $X$ and a bag $b$ of $X$. Now we state the propositions:

(23) $\overline{\overline{b}} = 0$ if and only if support $b = \emptyset$.

(24) $b$ is zero if and only if support $b = \emptyset$.

(25) $b$ is zero if and only if rng $b = \{0\}$.

Let $X$ be a non empty set, $b_1$ be a non zero bag of $X$, and $b_2$ be a bag of $X$. One can check that $b_1 + b_2$ is non zero.

(26) Let us consider a non empty set $X$, a bag $b$ of $X$, and an element $x$ of $X$. Suppose support $b = \{x\}$. Then $b = (\{x\}, b(x))$-bag.

(27) Let us consider a non empty set $X$, a non empty bag $b$ of $X$, and an element $x$ of $X$. Then support $b = \{x\}$ if and only if $b = (\{x\}, b(x))$-bag and $b(x) \neq 0$. The theorem is a consequence of (26).

Let $X$ be a set and $S$ be a finite subset of $X$. The functor Bag$(S)$ yielding a bag of $X$ is defined by the term

(Def. 1) $(S, 1)$-bag.

Let $X$ be a non empty set and $S$ be a non empty, finite subset of $X$. Observe that Bag$(S)$ is non zero.

Let $b$ be a bag of $X$ and $a$ be an element of $X$. The functor $b \setminus a$ yielding a bag of $X$ is defined by the term

(Def. 2) $b +\cdot (a, 0)$.

Let us consider a non empty set $X$, a bag $b$ of $X$, and an element $a$ of $X$. Now we state the propositions:

(28) $b \setminus a = b$ if and only if $a \notin$ support $b$.

(29) support$(b \setminus a) =$ support $b \setminus \{a\}$.

(30) $(b \setminus a) + (\{a\}, b(a))$-bag $= b$.

(31) Let us consider a non empty set $X$, an element $a$ of $X$, and an element $n$ of $\mathbb{N}$. Then $\overline{\overline{(\{a\}, n)\text{-bag}}} = n$. The theorem is a consequence of (23).

## 5. On Multiple Roots of Polynomials

Let $R$ be an integral domain and $p$ be a non zero polynomial over $R$ with roots. One can verify that BRoots$(p)$ is non zero.

Now we state the propositions:

(32) Let us consider a non degenerated commutative ring $R$, a non zero polynomial $p$ over $R$, and an element $a$ of $R$. Then multiplicity$(p, a) = 0$ if and only if rpoly$(1, a) \nmid p$.

(33) Let us consider an integral domain $R$, a non zero polynomial $p$ over $R$, and an element $a$ of $R$. Then $\mathrm{multiplicity}(p, a) = n$ if and only if $(\mathrm{rpoly}(1, a))^n \mid p$ and $(\mathrm{rpoly}(1, a))^{n+1} \nmid p$. The theorem is a consequence of (10).

(34) Let us consider an integral domain $R$, and an element $a$ of $R$. Then $\mathrm{multiplicity}(\mathrm{rpoly}(1, a), a) = 1$. The theorem is a consequence of (13) and (33).

(35) Let us consider an integral domain $R$, and elements $a$, $b$ of $R$. If $b \neq a$, then $\mathrm{multiplicity}(\mathrm{rpoly}(1, a), b) = 0$. The theorem is a consequence of (21) and (32).

(36) Let us consider an integral domain $R$, a non zero polynomial $p$ over $R$, a non zero element $b$ of $R$, and an element $a$ of $R$. Then $\mathrm{multiplicity}(p, a) = \mathrm{multiplicity}(b \cdot p, a)$. The theorem is a consequence of (33), (14), and (17).

(37) Let us consider an integral domain $R$, a non zero polynomial $p$ over $R$, and a non zero element $b$ of $R$. Then $\mathrm{BRoots}(b \cdot p) = \mathrm{BRoots}(p)$. The theorem is a consequence of (36).

(38) Let us consider an integral domain $R$, and a non zero polynomial $p$ over $R$ without roots. Then $\mathrm{BRoots}(p) = \mathrm{EmptyBag}(\text{the carrier of } R)$.

(39) Let us consider an integral domain $R$, and a non zero element $a$ of $R$. Then $\overline{\overline{\mathrm{BRoots}(a \restriction R)}} = 0$. The theorem is a consequence of (23).

(40) Let us consider an integral domain $R$, and an element $a$ of $R$. Then $\overline{\overline{\mathrm{BRoots}(\mathrm{rpoly}(1, a))}} = 1$. The theorem is a consequence of (10).

(41) Let us consider an integral domain $R$, and non zero polynomials $p$, $q$ over $R$. Then $\overline{\overline{\mathrm{BRoots}(p * q)}} = \overline{\overline{\mathrm{BRoots}(p)}} + \overline{\overline{\mathrm{BRoots}(q)}}$.

(42) Let us consider an integral domain $R$, and a non zero polynomial $p$ over $R$. Then $\overline{\overline{\mathrm{BRoots}(p)}} \leqslant \deg p$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every non zero polynomial $p$ over $R$ such that $\deg p = \$_1$ holds $\overline{\overline{\mathrm{BRoots}(p)}} \leqslant \deg p$. $\mathcal{P}[0]$. For every natural number $k$, $\mathcal{P}[k]$. □

## 6. The Polynomial $X^n + 1$

Let $R$ be a unital, non empty double loop structure and $n$ be a natural number. The functor npoly$(R, n)$ yielding a sequence of $R$ is defined by the term

(Def. 3) $\mathbf{0}.R{+\cdot}[0 \longmapsto 1_R, n \longmapsto 1_R]$.

One can check that npoly$(R, n)$ is finite-Support and npoly$(R, n)$ is non zero.

Let us consider a unital, non degenerated double loop structure $R$. Now we state the propositions:

(43) $\deg \mathrm{npoly}(R, n) = n$.

(44) $\mathrm{LC}\, \mathrm{npoly}(R, n) = 1_R$.

(45) Let us consider a non degenerated ring $R$, and an element $x$ of $R$. Then $\mathrm{eval}(\mathrm{npoly}(R, 0), x) = 1_R$.

(46) Let us consider a non degenerated ring $R$, a non zero natural number $n$, and an element $x$ of $R$. Then $\mathrm{eval}(\mathrm{npoly}(R, n), x) = x^n + 1_R$.

Proof: Set $q = \mathrm{npoly}(R, n)$. Consider $F$ being a finite sequence of elements of $R$ such that $\mathrm{eval}(q, x) = \sum F$ and $\mathrm{len}\, F = \mathrm{len}\, q$ and for every element $j$ of $\mathbb{N}$ such that $j \in \mathrm{dom}\, F$ holds $F(j) = q(j -' 1) \cdot \mathrm{power}_R(x, j -' 1)$. Consider $f_1$ being a sequence of the carrier of $R$ such that $\sum F = f_1(\mathrm{len}\, F)$ and $f_1(0) = 0_R$ and for every natural number $j$ and for every element $v$ of $R$ such that $j < \mathrm{len}\, F$ and $v = F(j + 1)$ holds $f_1(j + 1) = f_1(j) + v$. Define $\mathcal{P}[\text{element of } \mathbb{N}] \equiv \$_1 = 0$ and $f_1(\$_1) = 0_R$ or $0 < \$_1 < \mathrm{len}\, F$ and $f_1(\$_1) = 1_R$ or $\$_1 = \mathrm{len}\, F$ and $f_1(\$_1) = x^n + 1_R$. For every element $j$ of $\mathbb{N}$ such that $0 \leqslant j \leqslant \mathrm{len}\, F$ holds $\mathcal{P}[j]$. □

(47) Let us consider an even natural number $n$, and an element $x$ of $\mathbb{R}_{\mathrm{F}}$. Then $\mathrm{eval}(\mathrm{npoly}(\mathbb{R}_{\mathrm{F}}, n), x) > 0_{\mathbb{R}_{\mathrm{F}}}$. The theorem is a consequence of (45), (1), and (46).

(48) Let us consider an odd natural number $n$. Then $\mathrm{eval}(\mathrm{npoly}(\mathbb{R}_{\mathrm{F}}, n), -1_{\mathbb{R}_{\mathrm{F}}}) = 0_{\mathbb{R}_{\mathrm{F}}}$. The theorem is a consequence of (46).

(49) $\mathrm{eval}(\mathrm{npoly}(\mathbb{Z}/2, 2), 1_{\mathbb{Z}/2}) = 0_{\mathbb{Z}/2}$. The theorem is a consequence of (46) and (2).

Let $n$ be an even natural number. Let us note that npoly$(\mathbb{R}_{\mathrm{F}}, n)$ and has not roots.

Let $n$ be an odd natural number. Observe that npoly$(\mathbb{R}_{\mathrm{F}}, n)$ has roots and npoly$(\mathbb{Z}/2, 2)$ has roots.

## 7. The Polynomials $(x - a_1) * (x - a_2) * \ldots * (x - a_n)$

Let $R$ be a ring.

A product of linear polynomials of $R$ is a polynomial over $R$ and is defined by

(Def. 4) there exists a non empty finite sequence $F$ of elements of PolyRing($R$) such that $it = \prod F$ and for every natural number $i$ such that $i \in \operatorname{dom} F$ there exists an element $a$ of $R$ such that $F(i) = \operatorname{rpoly}(1, a)$.

Let $R$ be an integral domain. One can verify that every product of linear polynomials of $R$ is non constant and monic and has roots.

Now we state the propositions:

(50) Let us consider an integral domain $R$, and a product of linear polynomials $p$ of $R$. Then $\operatorname{LC} p = 1_R$.

(51) Let us consider an integral domain $R$, and an element $a$ of $R$. Then $\operatorname{rpoly}(1, a)$ is a product of linear polynomials of $R$.

(52) Let us consider an integral domain $R$, and products of linear polynomials $p$, $q$ of $R$. Then $p * q$ is a product of linear polynomials of $R$.

Let $R$ be an integral domain and $B$ be a non zero bag of the carrier of $R$.

A product of linear polynomials of $R$ and $B$ is a product of linear polynomials of $R$ and is defined by

(Def. 5) $\deg it = \overline{\overline{B}}$ and for every element $a$ of $R$, $\operatorname{multiplicity}(it, a) = B(a)$.

Let us consider an integral domain $R$, a non zero bag $B$ of the carrier of $R$, a product of linear polynomials $p$ of $R$ and $B$, and an element $a$ of $R$. Now we state the propositions:

(53) If $a \in \operatorname{support} B$, then $\operatorname{eval}(p, a) = 0_R$. The theorem is a consequence of (11).

(54) (i) $(\operatorname{rpoly}(1, a))^{B(a)} \mid p$, and

(ii) $(\operatorname{rpoly}(1, a))^{B(a)+1} \nmid p$.

The theorem is a consequence of (33).

Let us consider an integral domain $R$, a non zero bag $B$ of the carrier of $R$, and a product of linear polynomials $p$ of $R$ and $B$. Now we state the propositions:

(55) $\operatorname{BRoots}(p) = B$.

(56) $\deg p = \overline{\overline{\operatorname{BRoots}(p)}}$. The theorem is a consequence of (55).

(57) Let us consider an integral domain $R$, and an element $a$ of $R$. Then $\operatorname{rpoly}(1, a)$ is a product of linear polynomials of $R$ and $\operatorname{Bag}(\{a\})$. The theorem is a consequence of (51), (34), and (35).

(58) Let us consider an integral domain $R$, non zero bags $B_1$, $B_2$ of the carrier of $R$, a product of linear polynomials $p$ of $R$ and $B_1$, and a product of linear

polynomials $q$ of $R$ and $B_2$. Then $p * q$ is a product of linear polynomials of $R$ and $B_1 + B_2$. The theorem is a consequence of (52), (56), and (55).

(59) Let us consider an integral domain $R$. Then every product of linear polynomials of $R$ is a product of linear polynomials of $R$ and BRoots($p$).
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every product of linear polynomials $p$ of $R$ such that $\deg p = \$_1$ holds $p$ is a product of linear polynomials of $R$ and BRoots($p$). $\mathcal{P}[1]$. For every natural number $k$ such that $k \geqslant 1$ holds $\mathcal{P}[k]$. □

Let $R$ be an integral domain and $S$ be a non empty, finite subset of $R$.

A product of linear polynomials of $R$ and $S$ is a product of linear polynomials of $R$ and Bag($S$). Now we state the proposition:

(60) Let us consider an integral domain $R$, a non empty, finite subset $S$ of $R$, and a product of linear polynomials $p$ of $R$ and $S$. Then $\deg p = \overline{\overline{S}}$.

Let us consider an integral domain $R$, a non empty, finite subset $S$ of $R$, a product of linear polynomials $p$ of $R$ and $S$, and an element $a$ of $R$. Now we state the propositions:

(61) If $a \in S$, then $\mathrm{rpoly}(1, a) \mid p$ and $(\mathrm{rpoly}(1, a))^2 \nmid p$. The theorem is a consequence of (54).

(62) If $a \in S$, then $\mathrm{eval}(p, a) = 0_R$. The theorem is a consequence of (61).

(63) Let us consider an integral domain $R$, a non empty, finite subset $S$ of $R$, and a product of linear polynomials $p$ of $R$ and $S$. Then Roots($p$) $= S$. The theorem is a consequence of (62), (22), and (60).

## 8. Main Theorems

Now we state the proposition:

(64) Let us consider an integral domain $R$, and a non zero polynomial $p$ over $R$ with roots. Then there exists a product of linear polynomials $q$ of $R$ and BRoots($p$) and there exists a polynomial $r$ over $R$ with non roots such that $p = q * r$ and Roots($q$) $=$ Roots($p$).
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every non zero polynomial $p$ over $R$ with roots such that $\deg p = \$_1$ there exists a product of linear polynomials $q$ of $R$ and BRoots($p$) and there exists a polynomial $r$ over $R$ with non roots such that $p = q * r$ and Roots($q$) $=$ Roots($p$). $\mathcal{P}[1]$ by (11), [9, (1)], (51), [8, (23), (27), (24)]. For every natural number $k$ such that $1 \leqslant k$ holds $\mathcal{P}[k]$. Consider $d$ being a natural number such that $\deg p = d$. □

Let us consider an integral domain $R$ and a non zero polynomial $p$ over $R$.

(65) $\overline{\overline{\mathrm{Roots}(p)}} \leqslant \overline{\overline{\mathrm{BRoots}(p)}}$. The theorem is a consequence of (64), (56), (55), (22), and (38).

(66) $\overline{\overline{\mathrm{BRoots}(p)}} = \deg p$ if and only if there exists an element $a$ of $R$ and there exists a product of linear polynomials $q$ of $R$ such that $p = a \cdot q$. The theorem is a consequence of (64), (56), (55), (59), (4), (37), and (38).

Now we state the proposition:

(67) Let us consider an integral domain $R$, and polynomials $p$, $q$ over $R$. Suppose there exists a subset $S$ of $R$ such that $\overline{\overline{S}} = \max(\deg p, \deg q) + 1$ and for every element $a$ of $R$ such that $a \in S$ holds $\mathrm{eval}(p, a) = \mathrm{eval}(q, a)$. Then $p = q$. The theorem is a consequence of (22).

Let $F$ be an algebraic closed field. Note that every non constant polynomial over $F$ has roots and $\mathbb{R}_{\mathrm{F}}$ is non algebraic closed and every finite integral domain is non algebraic closed and every ring which is algebraic closed is also almost right invertible.

Now we state the propositions:

(68) Let us consider an algebraic closed field $F$, and a non constant polynomial $p$ over $F$. Then there exists an element $a$ of $F$ and there exists a product of linear polynomials $q$ of $F$ and $\mathrm{BRoots}(p)$ such that $a \cdot q = p$. The theorem is a consequence of (64).

(69) Let us consider an algebraic closed field $F$. Then every non constant, monic polynomial over $F$ is a product of linear polynomials of $F$ and $\mathrm{BRoots}(p)$. The theorem is a consequence of (68).

(70) Let us consider a field $F$. Then $F$ is algebraic closed if and only if every non constant, monic polynomial over $F$ is a product of linear polynomials of $F$. The theorem is a consequence of (69).

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