FORMALIZED MATHEMATICS
Vol. 25, No. 3, Pages 227–240, 2017
DOI: 10.1515/forma-2017-0022

DE GRUYTER OPEN
degruyter.com/view/j/forma

Integral of Non Positive Functions

Noboru Endou
National Institute of Technology, Gifu College
2236-2 Kamimakuwa, Motosu, Gifu, Japan

Summary. In this article, we formalize in the Mizar system [1, 7] the Lebesgue type integral and convergence theorems for non positive functions [8],[2]. Many theorems are based on our previous results [5], [6].

MSC: 28A25 03B35

Keywords: integration of non positive function

MML identifier: MESFUN11, version: 8.1.06 5.44.1305

1. Preliminaries

Let X be a non empty set and f be a non-negative partial function from X to $\overline{\mathbb{R}}$. Observe that $-f$ is non-positive.

Let f be a non-positive partial function from X to $\overline{\mathbb{R}}$. One can check that $-f$ is non-negative.

Now we state the propositions:

(1) Let us consider a non empty set X, a non-positive partial function f from X to $\overline{\mathbb{R}}$, and a set E. Then $f \restriction E$ is non-positive.

(2) Let us consider a non empty set X, a set A, a real number r, and a partial function f from X to $\overline{\mathbb{R}}$. Then $(r \cdot f) \restriction A = r \cdot (f \restriction A)$.

(3) Let us consider a non empty set X, a set A, and a partial function f from X to $\overline{\mathbb{R}}$. Then $-f \restriction A = (-f) \restriction A$. The theorem is a consequence of (2).

(4) Let us consider a non empty set X, a partial function f from X to $\overline{\mathbb{R}}$, and a real number c. Suppose f is non-positive. Then

(i) if $0 \leqslant c$, then $c \cdot f$ is non-positive, and

(ii) if $c \leqslant 0$, then $c \cdot f$ is non-negative.

© 2017 University of Białystok
CC-BY-SA License ver. 3.0 or later
ISSN 1426–2630(Print), 1898-9934(Online)

(5) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a partial function f from X to $\overline{\mathbb{R}}$. Then

(i) $\max_+(f)$ is non-negative, and

(ii) $\max_-(f)$ is non-negative, and

(iii) $|f|$ is non-negative.

(6) Let us consider a non empty set X, a partial function f from X to $\overline{\mathbb{R}}$, and an object x. Then

(i) $f(x) \leqslant (\max_+(f))(x)$, and

(ii) $f(x) \geqslant -(\max_-(f))(x)$.

(7) Let us consider a non empty set X, a partial function f from X to $\overline{\mathbb{R}}$, and a positive real number r. Then $\text{LE-dom}(f, r) = \text{LE-dom}(\max_+(f), r)$.

(8) Let us consider a non empty set X, a partial function f from X to $\overline{\mathbb{R}}$, and a non positive real number r. Then $\text{LE-dom}(f, r) = \text{GT-dom}(\max_-(f), -r)$.

(9) Let us consider a non empty set X, partial functions f, g from X to $\overline{\mathbb{R}}$, an extended real a, and a real number r. Suppose $r \neq 0$ and $g = r \cdot f$. Then $\text{EQ-dom}(f, a) = \text{EQ-dom}(g, a \cdot r)$.

(10) Let us consider a non empty set X, a σ-field S of subsets of X, a partial function f from X to $\overline{\mathbb{R}}$, and an element A of S. Suppose $A \subseteq \text{dom}\, f$. Then f is measurable on A if and only if $\max_+(f)$ is measurable on A and $\max_-(f)$ is measurable on A.

Let X be a non empty set, f be a function from X into $\overline{\mathbb{R}}$, and r be a real number. Note that the functor $r \cdot f$ yields a function from X into $\overline{\mathbb{R}}$. Now we state the proposition:

(11) Let us consider a non empty set X, a real number r, and a without $+\infty$ function f from X into $\overline{\mathbb{R}}$. If $r \geqslant 0$, then $r \cdot f$ is without $+\infty$.

Let X be a non empty set, f be a without $+\infty$ function from X into $\overline{\mathbb{R}}$, and r be a non negative real number. Let us note that $r \cdot f$ is without $+\infty$ as a function from X into $\overline{\mathbb{R}}$.

Now we state the proposition:

(12) Let us consider a non empty set X, a real number r, and a without $+\infty$ function f from X into $\overline{\mathbb{R}}$. If $r \leqslant 0$, then $r \cdot f$ is without $-\infty$.

Let X be a non empty set, f be a without $+\infty$ function from X into $\overline{\mathbb{R}}$, and r be a non positive real number. One can check that $r \cdot f$ is without $-\infty$.

Now we state the proposition:

(13) Let us consider a non empty set X, a real number r, and a without $-\infty$ function f from X into $\overline{\mathbb{R}}$. If $r \geqslant 0$, then $r \cdot f$ is without $-\infty$.

Let X be a non empty set, f be a without $-\infty$ function from X into $\overline{\mathbb{R}}$, and r be a non negative real number. One can check that $r \cdot f$ is without $-\infty$.

Now we state the proposition:

(14) Let us consider a non empty set X, a real number r, and a without $-\infty$ function f from X into $\overline{\mathbb{R}}$. If $r \leqslant 0$, then $r \cdot f$ is without $+\infty$.

Let X be a non empty set, f be a without $-\infty$ function from X into $\overline{\mathbb{R}}$, and r be a non positive real number. One can check that $r \cdot f$ is without $+\infty$.

Now we state the proposition:

(15) Let us consider a non empty set X, a real number r, and a without $-\infty$, without $+\infty$ function f from X into $\overline{\mathbb{R}}$. Then $r \cdot f$ is without $-\infty$ and without $+\infty$.

Let X be a non empty set, f be a without $-\infty$, without $+\infty$ function from X into $\overline{\mathbb{R}}$, and r be a real number. Note that $r \cdot f$ is without $-\infty$ and without $+\infty$.

Now we state the propositions:

(16) Let us consider a non empty set X, a positive real number r, and a function f from X into $\overline{\mathbb{R}}$. Then f is without $+\infty$ if and only if $r \cdot f$ is without $+\infty$.

(17) Let us consider a non empty set X, a negative real number r, and a function f from X into $\overline{\mathbb{R}}$. Then f is without $+\infty$ if and only if $r \cdot f$ is without $-\infty$.

(18) Let us consider a non empty set X, a positive real number r, and a function f from X into $\overline{\mathbb{R}}$. Then f is without $-\infty$ if and only if $r \cdot f$ is without $-\infty$.

(19) Let us consider a non empty set X, a negative real number r, and a function f from X into $\overline{\mathbb{R}}$. Then f is without $-\infty$ if and only if $r \cdot f$ is without $+\infty$.

(20) Let us consider a non empty set X, a non zero real number r, and a function f from X into $\overline{\mathbb{R}}$. Then f is without $-\infty$ and without $+\infty$ if and only if $r \cdot f$ is without $-\infty$ and without $+\infty$. The theorem is a consequence of (16), (18), (17), and (19).

(21) Let us consider non empty sets X, Y, a partial function f from X to $\overline{\mathbb{R}}$, and a real number r. Suppose $f = Y \longmapsto r$. Then f is without $-\infty$ and without $+\infty$.

(22) Let us consider a non empty set X, and a function f from X into $\overline{\mathbb{R}}$. Then

(i) $0 \cdot f = X \longmapsto 0$, and

(ii) $0 \cdot f$ is without $-\infty$ and without $+\infty$.

PROOF: For every element x of X, $(0 \cdot f)(x) = (X \longmapsto 0)(x)$. □

(23) Let us consider a non empty set X, and partial functions f, g from X to $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ and without $+\infty$. Then

(i) $\operatorname{dom}(f + g) = \operatorname{dom} f \cap \operatorname{dom} g$, and

(ii) $\operatorname{dom}(f - g) = \operatorname{dom} f \cap \operatorname{dom} g$, and

(iii) $\operatorname{dom}(g - f) = \operatorname{dom} f \cap \operatorname{dom} g$.

Let us consider a non empty set X and functions f_1, f_2 from X into $\overline{\mathbb{R}}$. Now we state the propositions:

(24) Suppose f_2 is without $-\infty$ and without $+\infty$. Then

(i) $f_1 + f_2$ is a function from X into $\overline{\mathbb{R}}$, and

(ii) for every element x of X, $(f_1 + f_2)(x) = f_1(x) + f_2(x)$.

The theorem is a consequence of (23).

(25) Suppose f_1 is without $-\infty$ and without $+\infty$. Then

(i) $f_1 - f_2$ is a function from X into $\overline{\mathbb{R}}$, and

(ii) for every element x of X, $(f_1 - f_2)(x) = f_1(x) - f_2(x)$.

The theorem is a consequence of (23).

(26) Suppose f_2 is without $-\infty$ and without $+\infty$. Then

(i) $f_1 - f_2$ is a function from X into $\overline{\mathbb{R}}$, and

(ii) for every element x of X, $(f_1 - f_2)(x) = f_1(x) - f_2(x)$.

The theorem is a consequence of (23).

(27) Let us consider non empty sets X, Y, and partial functions f_1, f_2 from X to $\overline{\mathbb{R}}$. Suppose $\operatorname{dom} f_1 \subseteq Y$ and $f_2 = Y \longmapsto 0$. Then

(i) $f_1 + f_2 = f_1$, and

(ii) $f_1 - f_2 = f_1$, and

(iii) $f_2 - f_1 = -f_1$.

The theorem is a consequence of (21) and (23).

Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and partial functions f, g from X to $\overline{\mathbb{R}}$. Now we state the propositions:

(28) If f is simple function in S and g is simple function in S, then $f + g$ is simple function in S.

PROOF: Consider F being a finite sequence of separated subsets of S, a being a finite sequence of elements of $\overline{\mathbb{R}}$ such that F and a are representation of f. Consider G being a finite sequence of separated subsets of S, b being a finite sequence of elements of $\overline{\mathbb{R}}$ such that G and b are representation of g. Set $l_1 = \operatorname{len} a$. Set $l_2 = \operatorname{len} b$. Define $\mathcal{H}(\text{natural number}) =$

$F((\$_1 -' 1 \operatorname{div} l_2) + 1) \cap G((\$_1 -' 1 \bmod l_2) + 1)$. Consider F_1 being a finite sequence such that $\operatorname{len} F_1 = l_1 \cdot l_2$ and for every natural number k such that $k \in \operatorname{dom} F_1$ holds $F_1(k) = \mathcal{H}(k)$. For every natural numbers k, l such that k, $l \in \operatorname{dom} F_1$ and $k \neq l$ holds $F_1(k)$ misses $F_1(l)$. $\operatorname{dom}(f+g) = \bigcup \operatorname{rng} F_1$. For every natural number k and for every elements x, y of X such that $k \in \operatorname{dom} F_1$ and x, $y \in F_1(k)$ holds $(f+g)(x) = (f+g)(y)$. □

(29) If f is simple function in S and g is simple function in S, then $f - g$ is simple function in S. The theorem is a consequence of (28).

(30) Let us consider a non empty set X, a σ-field S of subsets of X, and a partial function f from X to $\overline{\mathbb{R}}$. If f is simple function in S, then $-f$ is simple function in S.

(31) Let us consider a non empty set X, and a non-negative partial function f from X to $\overline{\mathbb{R}}$. Then $f = \max_+(f)$.

Proof: For every element x of X such that $x \in \operatorname{dom} f$ holds $f(x) = (\max_+(f))(x)$. □

(32) Let us consider a non empty set X, and a non-positive partial function f from X to $\overline{\mathbb{R}}$. Then $f = -\max_-(f)$.

Proof: For every element x of X such that $x \in \operatorname{dom} f$ holds $f(x) = (-\max_-(f))(x)$. □

(33) Let us consider a non empty set C, a partial function f from C to $\overline{\mathbb{R}}$, and a real number c. Suppose $c \leqslant 0$. Then

(i) $\max_+(c \cdot f) = (-c) \cdot \max_-(f)$, and

(ii) $\max_-(c \cdot f) = (-c) \cdot \max_+(f)$.

Proof: For every element x of C such that $x \in \operatorname{dom} \max_+(c \cdot f)$ holds $(\max_+(c \cdot f))(x) = ((-c) \cdot \max_-(f))(x)$. For every element x of C such that $x \in \operatorname{dom} \max_-(c \cdot f)$ holds $(\max_-(c \cdot f))(x) = ((-c) \cdot \max_+(f))(x)$. □

(34) Let us consider a non empty set X, and a partial function f from X to $\overline{\mathbb{R}}$. Then $\max_+(f) = \max_-(-f)$. The theorem is a consequence of (33).

(35) Let us consider a non empty set X, a partial function f from X to $\overline{\mathbb{R}}$, and real numbers r_1, r_2. Then $r_1 \cdot (r_2 \cdot f) = (r_1 \cdot r_2) \cdot f$.

(36) Let us consider a non empty set X, and partial functions f, g from X to $\overline{\mathbb{R}}$. If $f = -g$, then $g = -f$. The theorem is a consequence of (35).

Let X be a non empty set, F be a sequence of partial functions from X into $\overline{\mathbb{R}}$, and r be a real number. The functor $r \cdot F$ yielding a sequence of partial functions from X into $\overline{\mathbb{R}}$ is defined by

(Def. 1) for every natural number n, $it(n) = r \cdot F(n)$.

The functor $-F$ yielding a sequence of partial functions from X into $\overline{\mathbb{R}}$ is defined by the term

(Def. 2) $(-1) \cdot F$.

Now we state the proposition:

(37) Let us consider a non empty set X, a sequence F of partial functions from X into $\overline{\mathbb{R}}$, and a natural number n. Then $(-F)(n) = -F(n)$.

Let us consider a non empty set X, a sequence F of partial functions from X into $\overline{\mathbb{R}}$, and an element x of X. Now we state the propositions:

(38) $(-F)\#x = -F\#x$. The theorem is a consequence of (37).

(39) (i) $F\#x$ is convergent to $+\infty$ iff $(-F)\#x$ is convergent to $-\infty$, and

(ii) $F\#x$ is convergent to $-\infty$ iff $(-F)\#x$ is convergent to $+\infty$, and

(iii) $F\#x$ is convergent to a finite limit iff $(-F)\#x$ is convergent to a finite limit, and

(iv) $F\#x$ is convergent iff $(-F)\#x$ is convergent, and

(v) if $F\#x$ is convergent, then $\lim((-F)\#x) = -\lim(F\#x)$.

The theorem is a consequence of (38).

Let us consider a non empty set X and a sequence F of partial functions from X into $\overline{\mathbb{R}}$. Now we state the propositions:

(40) If F has the same dom, then $-F$ has the same dom. The theorem is a consequence of (37).

(41) If F is additive, then $-F$ is additive. The theorem is a consequence of (37).

(42) Let us consider a non empty set X, a sequence F of partial functions from X into $\overline{\mathbb{R}}$, and a natural number n. Then $(\sum_{\alpha=0}^{\kappa}(-F)(\alpha))_{\kappa\in\mathbb{N}}(n) = (-(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})(n)$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa}(-F)(\alpha))_{\kappa\in\mathbb{N}}(\$_1) = (-(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})(\$_1)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(43) Let us consider a sequence s of extended reals, and a natural number n. Then $(\sum_{\alpha=0}^{\kappa}(-s)(\alpha))_{\kappa\in\mathbb{N}}(n) = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n)$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa}(-s)(\alpha))_{\kappa\in\mathbb{N}}(\$_1) = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(\$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

Let us consider a sequence s of extended reals. Now we state the propositions:

(44) $(\sum_{\alpha=0}^{\kappa}(-s)(\alpha))_{\kappa\in\mathbb{N}} = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$. The theorem is a consequence of (43).

(45) If s is summable, then $-s$ is summable. The theorem is a consequence of (44).

Let us consider a non empty set X and a sequence F of partial functions from X into $\overline{\mathbb{R}}$. Now we state the propositions:

(46) If for every natural number n, $F(n)$ is without $+\infty$, then F is additive.

(47) If for every natural number n, $F(n)$ is without $-\infty$, then F is additive.

(48) Let us consider a non empty set X, a sequence F of partial functions from X into $\overline{\mathbb{R}}$, and an element x of X. Suppose $F\#x$ is summable. Then

(i) $(-F)\#x$ is summable, and

(ii) $\sum((-F)\#x) = -\sum(F\#x)$.

The theorem is a consequence of (45), (38), and (44).

(49) Let us consider a non empty set X, a σ-field S of subsets of X, and a sequence F of partial functions from X into $\overline{\mathbb{R}}$. Suppose F is additive and has the same dom and for every element x of X such that $x \in \mathrm{dom}(F(0))$ holds $F\#x$ is summable. Then $\lim(\sum_{\alpha=0}^{\kappa}(-F)(\alpha))_{\kappa\in\mathbb{N}} = -\lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}$.
Proof: Set $G = -F$. For every element n of $\mathbb{N}$, $(\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa\in\mathbb{N}}(n) = (-(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})(n)$. For every element x of X such that $x \in \mathrm{dom}\lim (\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa\in\mathbb{N}}$ holds $(\lim(\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa\in\mathbb{N}})(x) = (-\lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})(x)$. □

(50) Let us consider a non empty set X, a σ-field S of subsets of X, sequences F, G of partial functions from X into $\overline{\mathbb{R}}$, and an element E of S. Suppose $E \subseteq \mathrm{dom}(F(0))$ and F is additive and has the same dom and for every natural number n, $G(n) = F(n)\restriction E$. Then $\lim(\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa\in\mathbb{N}} = \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}\restriction E$.
Proof: For every element x of X such that $x \in E$ holds $F\#x = G\#x$. Set $P_1 = (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}$. Set $P_2 = (\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa\in\mathbb{N}}$. For every element x of X such that $x \in \mathrm{dom}\lim P_2$ holds $(\lim P_2)(x) = (\lim P_1)(x)$. For every element x of X such that $x \in \mathrm{dom}(\lim P_2\restriction E)$ holds $(\lim P_2\restriction E)(x) = (\lim P_1\restriction E)(x)$. □

2. Integral of Non Positive Measurable Functions

Now we state the propositions:

(51) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a non-negative partial function f from X to $\overline{\mathbb{R}}$. Then $\int' \max_{-}(-f)\,\mathrm{d}M = \int' f\,\mathrm{d}M$. The theorem is a consequence of (32), (36), and (35).

(52) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and an element A of S.

Suppose $A = \operatorname{dom} f$ and f is measurable on A. Then $\int -f \, \mathrm{d}M = -\int f \, \mathrm{d}M$. The theorem is a consequence of (36), (10), (5), and (34).

(53) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a non-negative partial function f from X to $\overline{\mathbb{R}}$, and an element E of S. Suppose $E = \operatorname{dom} f$ and f is measurable on E. Then

(i) $\int \max_{-}(f) \, \mathrm{d}M = 0$, and

(ii) $\int^{+} \max_{-}(f) \, \mathrm{d}M = 0$.

PROOF: $\max_{-}(f)$ is measurable on E. For every object x such that $x \in \operatorname{dom} \max_{-}(f)$ holds $(\max_{-}(f))(x) = 0$. □

Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and an element E of S. Now we state the propositions:

(54) If $E = \operatorname{dom} f$ and f is measurable on E, then $\int f \, \mathrm{d}M = \int \max_{+}(f) \, \mathrm{d}M - \int \max_{-}(f) \, \mathrm{d}M$. The theorem is a consequence of (10) and (5).

(55) If $E \subseteq \operatorname{dom} f$ and f is measurable on E, then $\int (-f) \restriction E \, \mathrm{d}M = -\int f \restriction E \, \mathrm{d}M$. The theorem is a consequence of (3) and (52).

(56) Let us consider a non empty set X, a σ-field S of subsets of X, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and (f **qua** extended real-valued function) is non-positive. Then there exists a sequence F of partial functions from X into $\overline{\mathbb{R}}$ such that

(i) for every natural number n, $F(n)$ is simple function in S and $\operatorname{dom}(F(n)) = \operatorname{dom} f$, and

(ii) for every natural number n, $F(n)$ is non-positive, and

(iii) for every natural numbers n, m such that $n \leqslant m$ for every element x of X such that $x \in \operatorname{dom} f$ holds $F(n)(x) \geqslant F(m)(x)$, and

(iv) for every element x of X such that $x \in \operatorname{dom} f$ holds $F \# x$ is convergent and $\lim(F \# x) = f(x)$.

The theorem is a consequence of (37), (30), and (39).

(57) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, and a non-positive partial function f from X to $\overline{\mathbb{R}}$. Suppose there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A. Then

(i) $\int f \, \mathrm{d}M = -\int^{+} \max_{-}(f) \, \mathrm{d}M$, and

(ii) $\int f \, \mathrm{d}M = -\int^{+} -f \, \mathrm{d}M$, and

(iii) $\int f \, \mathrm{d}M = -\int -f \, \mathrm{d}M$.

Proof: Consider A being an element of S such that $A = \operatorname{dom} f$ and f is measurable on A. $f = -\max_-(f)$. $-f = \max_-(f)$. For every element x of X such that $x \in \operatorname{dom} \max_+(f)$ holds $(\max_+(f))(x) = 0$. □

(58) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a non-positive partial function f from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S. Then

(i) $\int f \, dM = -\int' -f \, dM$, and

(ii) $\int f \, dM = -\int' \max_-(f) \, dM$.

The theorem is a consequence of (30), (57), (32), and (36).

Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and a real number c. Now we state the propositions:

(59) If f is simple function in S and f is non-negative, then $\int c \cdot f \, dM = c \cdot \int' f \, dM$.

(60) Suppose f is simple function in S and f is non-positive. Then

(i) $\int c \cdot f \, dM = -c \cdot \int' -f \, dM$, and

(ii) $\int c \cdot f \, dM = -(c \cdot \int' -f \, dM)$.

The theorem is a consequence of (35), (30), and (59).

(61) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose there exists an element A of S such that $A = \operatorname{dom} f$ and f is measurable on A and f is non-positive. Then $0 \geqslant \int f \, dM$. The theorem is a consequence of (57).

(62) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and elements A, B, E of S. Suppose $E = \operatorname{dom} f$ and f is measurable on E and f is non-positive and A misses B. Then $\int f{\restriction}(A \cup B) \, dM = \int f{\restriction}A \, dM + \int f{\restriction}B \, dM$. The theorem is a consequence of (3) and (52).

(63) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and elements A, E of S. Suppose $E = \operatorname{dom} f$ and f is measurable on E and f is non-positive. Then $0 \geqslant \int f{\restriction}A \, dM$. The theorem is a consequence of (61) and (1).

(64) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and elements A, B, E of S. Suppose $E = \operatorname{dom} f$ and f is measurable on E and f is non-positive and $A \subseteq B$. Then $\int f{\restriction}A \, dM \geqslant \int f{\restriction}B \, dM$. The theorem is a consequence of (3) and (52).

3. Convergence Theorems for Non Positive Function's Integration

Now we state the propositions:

(65) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose $E = \mathrm{dom}\, f$ and f is measurable on E and f is non-positive and $M(E \cap \text{EQ-dom}(f, -\infty)) \neq 0$. Then $\int f \,\mathrm{d}M = -\infty$. The theorem is a consequence of (9) and (52).

(66) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, and partial functions f, g from X to $\overline{\mathbb{R}}$. Suppose $E \subseteq \mathrm{dom}\, f$ and $E \subseteq \mathrm{dom}\, g$ and f is measurable on E and g is measurable on E and f is non-positive and for every element x of X such that $x \in E$ holds $g(x) \leqslant f(x)$. Then $\int g \restriction E \,\mathrm{d}M \leqslant \int f \restriction E \,\mathrm{d}M$. The theorem is a consequence of (3) and (52).

(67) Let us consider a non empty set X, a sequence F of partial functions from X into $\overline{\mathbb{R}}$, a σ-field S of subsets of X, an element E of S, and a natural number m. Suppose F has the same dom and $E = \mathrm{dom}(F(0))$ and for every natural number n, $F(n)$ is measurable on E and $F(n)$ is without $+\infty$. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$ is measurable on E. The theorem is a consequence of (37), (42), and (46).

(68) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a sequence F of partial functions from X into $\overline{\mathbb{R}}$, an element E of S, a sequence I of extended reals, and a natural number m. Suppose $E = \mathrm{dom}(F(0))$ and F is additive and has the same dom and for every natural number n, $F(n)$ is measurable on E and $F(n)$ is non-positive and $I(n) = \int F(n) \,\mathrm{d}M$. Then $\int (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m) \,\mathrm{d}M = (\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}}(m)$.

Proof: Set $G = -F$. Set $J = -I$. $G(0) = -F(0)$. G has the same dom. For every natural number n, $F(n)$ is measurable on E and $F(n)$ is without $+\infty$. For every natural number n, $G(n)$ is measurable on E and $G(n)$ is non-negative and $J(n) = \int G(n) \,\mathrm{d}M$. $\int (\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa \in \mathbb{N}}(m) \,\mathrm{d}M = (\sum_{\alpha=0}^{\kappa} J(\alpha))_{\kappa \in \mathbb{N}}(m)$. $\int (-(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})(m) \,\mathrm{d}M = (\sum_{\alpha=0}^{\kappa} J(\alpha))_{\kappa \in \mathbb{N}}(m)$.
$\int (-(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})(m) \,\mathrm{d}M = -(\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}}(m)$.
$\int -(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m) \,\mathrm{d}M = -(\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}}(m)$.
$-\int (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m) \,\mathrm{d}M = -(\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}}(m)$. □

(69) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a sequence F of partial functions from X into $\overline{\mathbb{R}}$, an element E of S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose $E \subseteq \mathrm{dom}\, f$ and f is non-positive and f is measurable on E and for every natural

number n, $F(n)$ is simple function in S and $F(n)$ is non-positive and $E \subseteq \mathrm{dom}(F(n))$ and for every element x of X such that $x \in E$ holds $F\#x$ is summable and $f(x) = \sum(F\#x)$. Then there exists a sequence I of extended reals such that

(i) for every natural number n, $I(n) = \int F(n)\restriction E \, \mathrm{d}M$, and

(ii) I is summable, and

(iii) $\int f\restriction E \, \mathrm{d}M = \sum I$.

PROOF: Set $g = -f$. Set $G = -F$. G is additive. For every natural number n, $G(n)$ is simple function in S and $G(n)$ is non-negative and $E \subseteq \mathrm{dom}(G(n))$. For every element x of X such that $x \in E$ holds $G\#x$ is summable and $g(x) = \sum(G\#x)$. Consider J being a sequence of extended reals such that for every natural number n, $J(n) = \int G(n)\restriction E \, \mathrm{d}M$ and J is summable and $\int g\restriction E \, \mathrm{d}M = \sum J$. For every natural number n, $I(n) = \int F(n)\restriction E \, \mathrm{d}M$. $\int g\restriction E \, \mathrm{d}M = -\int f\restriction E \, \mathrm{d}M$. $\lim(\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa \in \mathbb{N}} = -\int g\restriction E \, \mathrm{d}M$. □

(70) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, an element E of S, and a partial function f from X to $\overline{\mathbb{R}}$. Suppose $E \subseteq \mathrm{dom}\, f$ and f is non-positive and f is measurable on E. Then there exists a sequence F of partial functions from X into $\overline{\mathbb{R}}$ such that

(i) F is additive, and

(ii) for every natural number n, $F(n)$ is simple function in S and $F(n)$ is non-positive and $F(n)$ is measurable on E, and

(iii) for every element x of X such that $x \in E$ holds $F\#x$ is summable and $f(x) = \sum(F\#x)$, and

(iv) there exists a sequence I of extended reals such that for every natural number n, $I(n) = \int F(n)\restriction E \, \mathrm{d}M$ and I is summable and $\int f\restriction E \, \mathrm{d}M = \sum I$.

PROOF: Set $g = -f$. Consider G being a sequence of partial functions from X into $\overline{\mathbb{R}}$ such that G is additive and for every natural number n, $G(n)$ is simple function in S and $G(n)$ is non-negative and $G(n)$ is measurable on E and for every element x of X such that $x \in E$ holds $G\#x$ is summable and $g(x) = \sum(G\#x)$ and there exists a sequence J of extended reals such that for every natural number n, $J(n) = \int G(n)\restriction E \, \mathrm{d}M$ and J is summable and $\int g\restriction E \, \mathrm{d}M = \sum J$. For every natural number n, $F(n)$ is simple function in S and $F(n)$ is non-positive and $F(n)$ is measurable on E. For every element x of X such that $x \in E$ holds $F\#x$ is summable and $f(x) = \sum(F\#x)$. There exists a sequence I of extended reals such that for every natural number n, $I(n) = \int F(n){\restriction}E\,\mathrm{d}M$ and I is summable and $\int f{\restriction}E\,\mathrm{d}M = \sum I$. □

Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a sequence F of partial functions from X into $\overline{\mathbb{R}}$, and an element E of S. Now we state the propositions:

(71) Suppose $E = \mathrm{dom}(F(0))$ and F has the same dom and for every natural number n, $F(n)$ is non-positive and $F(n)$ is measurable on E. Then there exists a sequence F_1 of $(X \dot{\rightarrow} \overline{\mathbb{R}})^{\mathbb{N}}$ such that for every natural number n, for every natural number m, $F_1(n)(m)$ is simple function in S and $\mathrm{dom}(F_1(n)(m)) = \mathrm{dom}(F(n))$ and for every natural number m, $F_1(n)(m)$ is non-positive and for every natural numbers j, k such that $j \leqslant k$ for every element x of X such that $x \in \mathrm{dom}(F(n))$ holds $F_1(n)(j)(x) \geqslant F_1(n)(k)(x)$ and for every element x of X such that $x \in \mathrm{dom}(F(n))$ holds $F_1(n)\#x$ is convergent and $\lim(F_1(n)\#x) = F(n)(x)$.

PROOF: Define $\mathcal{Q}[\text{element of } \mathbb{N}, \text{set}] \equiv$ for every sequence G of partial functions from X into $\overline{\mathbb{R}}$ such that $\$_2 = G$ holds for every natural number m, $G(m)$ is simple function in S and $\mathrm{dom}(G(m)) = \mathrm{dom}(F(\$_1))$ and for every natural number m, $G(m)$ is non-positive and for every natural numbers j, k such that $j \leqslant k$ for every element x of X such that $x \in \mathrm{dom}(F(\$_1))$ holds $G(j)(x) \geqslant G(k)(x)$ and for every element x of X such that $x \in \mathrm{dom}(F(\$_1))$ holds $G\#x$ is convergent and $\lim(G\#x) = F(\$_1)(x)$. For every element n of $\mathbb{N}$, there exists a sequence G of partial functions from X into $\overline{\mathbb{R}}$ such that for every natural number m, $G(m)$ is simple function in S and $\mathrm{dom}(G(m)) = \mathrm{dom}(F(n))$ and for every natural number m, $G(m)$ is non-positive and for every natural numbers j, k such that $j \leqslant k$ for every element x of X such that $x \in \mathrm{dom}(F(n))$ holds $G(j)(x) \geqslant G(k)(x)$ and for every element x of X such that $x \in \mathrm{dom}(F(n))$ holds $G\#x$ is convergent and $\lim(G\#x) = F(n)(x)$. For every element n of $\mathbb{N}$, there exists an element G of $(X \dot{\rightarrow} \overline{\mathbb{R}})^{\mathbb{N}}$ such that $\mathcal{Q}[n, G]$. Consider F_1 being a sequence of $(X \dot{\rightarrow} \overline{\mathbb{R}})^{\mathbb{N}}$ such that for every element n of $\mathbb{N}$, $\mathcal{Q}[n, F_1(n)]$. For every natural number n, for every natural number m, $F_1(n)(m)$ is simple function in S and $\mathrm{dom}(F_1(n)(m)) = \mathrm{dom}(F(n))$ and for every natural number m, $F_1(n)(m)$ is non-positive and for every natural numbers j, k such that $j \leqslant k$ for every element x of X such that $x \in \mathrm{dom}(F(n))$ holds $F_1(n)(j)(x) \geqslant F_1(n)(k)(x)$ and for every element x of X such that $x \in \mathrm{dom}(F(n))$ holds $F_1(n)\#x$ is convergent and $\lim(F_1(n)\#x) = F(n)(x)$. □

(72) Suppose $E = \mathrm{dom}(F(0))$ and F is additive and has the same dom and for every natural number n, $F(n)$ is measurable on E and $F(n)$ is non-positive. Then there exists a sequence I of extended reals such that for every natural number n, $I(n) = \int F(n)\,\mathrm{d}M$ and $\int (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)\,\mathrm{d}M =$

$(\sum_{\alpha=0}^{\kappa} I(\alpha))_{\kappa\in\mathbb{N}}(n)$.

PROOF: Set $G = -F$. $G(0) = -F(0)$. G has the same dom. For every natural number n, $G(n)$ is measurable on E and $G(n)$ is non-negative. Consider J being a sequence of extended reals such that for every natural number n, $J(n) = \int G(n)\,\mathrm{d}M$ and $\int(\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa\in\mathbb{N}}(n)\,\mathrm{d}M = (\sum_{\alpha=0}^{\kappa} J(\alpha))_{\kappa\in\mathbb{N}}(n)$. For every natural number n, $F(n)$ is measurable on E and $F(n)$ is without $+\infty$. □

(73) Suppose $E \subseteq \mathrm{dom}(F(0))$ and F is additive and has the same dom and for every natural number n, $F(n)$ is non-positive and $F(n)$ is measurable on E and for every element x of X such that $x \in E$ holds $F\#x$ is summable. Then there exists a sequence I of extended reals such that

(i) for every natural number n, $I(n) = \int F(n)\restriction E\,\mathrm{d}M$, and

(ii) I is summable, and

(iii) $\int \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}\restriction E\,\mathrm{d}M = \sum I$.

PROOF: Set $G = -F$. $G(0) = -F(0)$. G is additive. G has the same dom. For every natural number n, $G(n)$ is non-negative and $G(n)$ is measurable on E. For every element x of X such that $x \in E$ holds $G\#x$ is summable. Consider J being a sequence of extended reals such that for every natural number n, $J(n) = \int G(n)\restriction E\,\mathrm{d}M$ and J is summable and $\int \lim(\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa\in\mathbb{N}}\restriction E\,\mathrm{d}M = \sum J$. For every natural number n, $I(n) = \int F(n)\restriction E\,\mathrm{d}M$. Define $\mathcal{H}(\text{natural number}) = F(\$_1)\restriction E$. Consider H being a sequence of partial functions from X into $\overline{\mathbb{R}}$ such that for every natural number n, $H(n) = \mathcal{H}(n)$. $\lim(\sum_{\alpha=0}^{\kappa} H(\alpha))_{\kappa\in\mathbb{N}} = \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}\restriction E$. Define $\mathcal{K}(\text{natural number}) = G(\$_1)\restriction E$. Consider K being a sequence of partial functions from X into $\overline{\mathbb{R}}$ such that for every natural number n, $K(n) = \mathcal{K}(n)$. $\lim(\sum_{\alpha=0}^{\kappa} K(\alpha))_{\kappa\in\mathbb{N}} = \lim(\sum_{\alpha=0}^{\kappa} G(\alpha))_{\kappa\in\mathbb{N}}\restriction E$. For every element n of $\mathbb{N}$, $H(n) = (-K)(n)$. $\lim(\sum_{\alpha=0}^{\kappa} H(\alpha))_{\kappa\in\mathbb{N}} =$
$-\lim(\sum_{\alpha=0}^{\kappa} K(\alpha))_{\kappa\in\mathbb{N}}$. For every natural number n, $K(n)$ is measurable on E and $K(n)$ is without $-\infty$. $\int(-\lim(\sum_{\alpha=0}^{\kappa} K(\alpha))_{\kappa\in\mathbb{N}})\restriction E\,\mathrm{d}M = -\int \lim(\sum_{\alpha=0}^{\kappa} K(\alpha))_{\kappa\in\mathbb{N}}\restriction E\,\mathrm{d}M$. □

(74) Suppose $E = \mathrm{dom}(F(0))$ and $F(0)$ is non-positive and F has the same dom and for every natural number n, $F(n)$ is measurable on E and for every natural numbers n, m such that $n \leqslant m$ for every element x of X such that $x \in E$ holds $F(n)(x) \geqslant F(m)(x)$ and for every element x of X such that $x \in E$ holds $F\#x$ is convergent. Then there exists a sequence I of extended reals such that

(i) for every natural number n, $I(n) = \int F(n)\,\mathrm{d}M$, and

(ii) I is convergent, and

(iii) $\int \lim F \, \mathrm{d}M = \lim I$.

Proof: Set $G = -F$. $G(0) = -F(0)$. For every natural number n, $G(n)$ is measurable on E by [4, (63)], (37). For every natural numbers n, m such that $n \leqslant m$ for every element x of X such that $x \in E$ holds $G(n)(x) \leqslant G(m)(x)$. For every element x of X such that $x \in E$ holds $G \# x$ is convergent. Consider J being a sequence of extended reals such that for every natural number n, $J(n) = \int G(n) \, \mathrm{d}M$ and J is convergent and $\int \lim G \, \mathrm{d}M = \lim J$. Set $I = -J$. For every natural number n, $I(n) = \int F(n) \, \mathrm{d}M$. For every element x of X such that $x \in \operatorname{dom} \lim G$ holds $(\lim G)(x) = (-\lim F)(x)$ by (38), [3, (17)]. $\int \lim G \, \mathrm{d}M = -\int \lim F \, \mathrm{d}M$. □

References

[1] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[2] Vladimir Igorevich Bogachev and Maria Aparecida Soares Ruas. *Measure theory*, volume 1. Springer, 2007.

[3] Noboru Endou. Extended real-valued double sequence and its convergence. *Formalized Mathematics*, 23(**3**):253–277, 2015. doi:10.1515/forma-2015-0021.

[4] Noboru Endou. Fubini's theorem on measure. *Formalized Mathematics*, 25(**1**):1–29, 2017. doi:10.1515/forma-2017-0001.

[5] Noboru Endou and Yasunari Shidama. Integral of measurable function. *Formalized Mathematics*, 14(**2**):53–70, 2006. doi:10.2478/v10037-006-0008-x.

[6] Noboru Endou, Keiko Narita, and Yasunari Shidama. The Lebesgue monotone convergence theorem. *Formalized Mathematics*, 16(**2**):167–175, 2008. doi:10.2478/v10037-008-0023-1.

[7] Adam Grabowski, Artur Korniłowicz, and Adam Naumowicz. Four decades of Mizar. *Journal of Automated Reasoning*, 55(3):191–198, 2015. doi:10.1007/s10817-015-9345-1.

[8] P. R. Halmos. *Measure Theory*. Springer-Verlag, 1974.

Received September 3, 2017

The English version of this volume of Formalized Mathematics was financed under agreement 548/P-DUN/2016 with the funds from the Polish Minister of Science and Higher Education for the dissemination of science.