FORMALIZED MATHEMATICS
Vol. 25, No. 2, Pages 107–119, 2017
DOI: 10.1515/forma-2017-0011

DE GRUYTER OPEN
degruyter.com/view/j/forma

# Pascal's Theorem in Real Projective Plane

Roland Coghetto
Rue de la Brasserie 5
7100 La Louvière, Belgium

**Summary.** In this article we check, with the Mizar system [2], Pascal's theorem in the real projective plane (in projective geometry Pascal's theorem is also known as the Hexagrammum Mysticum Theorem)[1]. Pappus' theorem is a special case of a degenerate conic of two lines.

For proving Pascal's theorem, we use the techniques developed in the section "Projective Proofs of Pappus' Theorem" in the chapter "Pappus' Theorem: Nine proofs and three variations" [11]. We also follow some ideas from Harrison's work. With HOL Light, he has the proof of Pascal's theorem[2]. For a lemma, we use PROVER9[3] and OTT2MIZ by Josef Urban[4] [12, 6, 7]. We note, that we don't use Skolem/Herbrand functions (see "Skolemization" in [1]).

MSC: 51E15 51N15 03B35

Keywords: Pascal's theorem; real projective plane; Grassman-Plücker relation

MML identifier: PASCAL, version: 8.1.06 5.43.1297

## 1. Preliminaries

From now on $n$ denotes a natural number, $K$ denotes a field, $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$, $i$, $a_1$, $b_1$, $c_1$, $d_1$, $e_1$, $f_1$, $g_1$, $h_1$, $i_1$ denote elements of $K$, $M$, $N$ denote square matrices over $K$ of dimension 3, and $p$ denotes a finite sequence of elements of $\mathbb{R}$.

Now we state the propositions:

(1) Let us consider points $p$, $q$, $r$ of $\mathcal{E}^3_{\mathrm{T}}$. Then

[1] https://en.wikipedia.org/wiki/Pascal's_theorem
[2] https://github.com/jrh13/hol-light/tree/master/100/pascal.ml
[3] https://www.cs.unm.edu/~mccune/prover9/
[4] https://github.com/JUrban/ott2miz




ISSN 1426–2630(Print), 1898-9934(Online)

(i) $\langle|p, q, r|\rangle = \langle|r, p, q|\rangle$, and

(ii) $\langle|p, q, r|\rangle = \langle|q, r, p|\rangle$.

(2) Suppose $\langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle = \langle\langle a_1, b_1, c_1\rangle, \langle d_1, e_1, f_1\rangle, \langle g_1, h_1, i_1\rangle\rangle$. Then

(i) $a = a_1$, and

(ii) $b = b_1$, and

(iii) $c = c_1$, and

(iv) $d = d_1$, and

(v) $e = e_1$, and

(vi) $f = f_1$, and

(vii) $g = g_1$, and

(viii) $h = h_1$, and

(ix) $i = i_1$.

(3) There exists $a$ and there exists $b$ and there exists $c$ and there exists $d$ and there exists $e$ and there exists $f$ and there exists $g$ and there exists $h$ and there exists $i$ such that $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$.

(4) Suppose $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$. Then

(i) $a = M_{1,1}$, and

(ii) $b = M_{1,2}$, and

(iii) $c = M_{1,3}$, and

(iv) $d = M_{2,1}$, and

(v) $e = M_{2,2}$, and

(vi) $f = M_{2,3}$, and

(vii) $g = M_{3,1}$, and

(viii) $h = M_{3,2}$, and

(ix) $i = M_{3,3}$.

(5) Suppose $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$. Then $M^{\mathrm{T}} = \langle\langle a, d, g\rangle, \langle b, e, h\rangle, \langle c, f, i\rangle\rangle$. The theorem is a consequence of (4) and (3).

(6) Suppose $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$ and $M$ is symmetric. Then

(i) $b = d$, and

(ii) $c = g$, and

(iii) $h = f$.

The theorem is a consequence of (5) and (2).

(7) Let us consider square matrices $M$, $N$ over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. If $N$ is symmetric, then $M^{\mathrm{T}} \cdot N \cdot M$ is symmetric.

(8) Let us consider a square matrix $M$ over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, elements $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$, $i$, $x$, $y$, $z$ of $\mathbb{R}_{\mathrm{F}}$, an element $v$ of $\mathcal{E}^3_{\mathrm{T}}$, a finite sequence $u_{10}$ of elements of $\mathbb{R}_{\mathrm{F}}$, and a finite sequence $p$ of elements of $\mathbb{R}^1$. Suppose $p = M \cdot u_{10}$ and $v = \mathrm{M2F}(p)$ and $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$ and $u_{10} = \langle x, y, z\rangle$. Then

(i) $p = \langle\langle a \cdot x + (b \cdot y) + (c \cdot z)\rangle, \langle d \cdot x + (e \cdot y) + (f \cdot z)\rangle, \langle g \cdot x + (h \cdot y) + (i \cdot z)\rangle\rangle$, and

(ii) $v = \langle a \cdot x + (b \cdot y) + (c \cdot z), d \cdot x + (e \cdot y) + (f \cdot z), g \cdot x + (h \cdot y) + (i \cdot z)\rangle$.

(9) Let us consider a square matrix $M$ over $\mathbb{R}$ of dimension 3, and elements $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$, $i$, $p_1$, $p_2$, $p_3$ of $\mathbb{R}$. Suppose $M = \langle\langle a, b, c\rangle, \langle d, e, f\rangle, \langle g, h, i\rangle\rangle$ and $p = \langle p_1, p_2, p_3\rangle$. Then $M \cdot p = \langle a \cdot p_1 + (b \cdot p_2) + (c \cdot p_3), d \cdot p_1 + (e \cdot p_2) + (f \cdot p_3), g \cdot p_1 + (h \cdot p_2) + (i \cdot p_3)\rangle$.

## 2. Conic in Real Projective Plane

Let $a$, $b$, $c$, $d$, $e$, $f$ be real numbers and $u$ be an element of $\mathcal{E}^3_{\mathrm{T}}$. The functor $\mathrm{qfconic}(a, b, c, d, e, f, u)$ yielding a real number is defined by the term

(Def. 1) $a \cdot u(1) \cdot u(1) + (b \cdot u(2) \cdot u(2)) + (c \cdot u(3) \cdot u(3)) + (d \cdot u(1) \cdot u(2)) + (e \cdot u(1) \cdot u(3)) + (f \cdot u(2) \cdot u(3))$.

The functor $\mathrm{conic}(a, b, c, d, e, f)$ yielding a subset of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ is defined by the term

(Def. 2) $\{P$, where $P$ is a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ : for every element $u$ of $\mathcal{E}^3_{\mathrm{T}}$ such that $u$ is not zero and $P =$ the direction of $u$ holds $\mathrm{qfconic}(a, b, c, d, e, f, u) = 0\}$.

In the sequel $a$, $b$, $c$, $d$, $e$, $f$ denote real numbers, $u$, $u_1$, $u_2$ denote non zero elements of $\mathcal{E}^3_{\mathrm{T}}$, and $P$ denotes an element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$.

Now we state the propositions:

(10) Suppose the direction of $u_1 =$ the direction of $u_2$ and $\mathrm{qfconic}(a, b, c, d, e, f, u_1) = 0$. Then $\mathrm{qfconic}(a, b, c, d, e, f, u_2) = 0$.

(11) If $P =$ the direction of $u$ and $\mathrm{qfconic}(a, b, c, d, e, f, u) = 0$, then $P \in \mathrm{conic}(a, b, c, d, e, f)$. The theorem is a consequence of (10).

Let $a$, $b$, $c$, $d$, $e$, $f$ be real numbers. The functor $\mathrm{symmetric3}(a, b, c, d, e, f)$ yielding a square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 is defined by the term

(Def. 3) $\langle\langle a, d, e\rangle, \langle d, b, f\rangle, \langle e, f, c\rangle\rangle$.

Now we state the propositions:

(12) symmetric3$(a, b, c, d, e, f)$ is symmetric. The theorem is a consequence of (5).

(13) Let us consider real numbers $a$, $b$, $c$, $d$, $e$, $f$, a point $u$ of $\mathcal{E}^3_{\mathrm{T}}$, and a square matrix $M$ over $\mathbb{R}$ of dimension 3. Suppose $p = u$ and $M =$ symmetric3$(a, b, c, d, e, f)$.
Then SumAll QuadraticForm$(p, M, p) =$ qfconic$(a, b, c, 2 \cdot d, 2 \cdot e, 2 \cdot f, u)$.

(14) Let us consider an invertible square matrix $N$ over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, square matrices $N_1$, $M_1$, $M_2$ over $\mathbb{R}$ of dimension 3, and real numbers $a$, $b$, $c$, $d$, $e$, $f$. Suppose $N_1 = (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})N$ and $M_1 =$ symmetric3$(a, b, c, \frac{d}{2}, \frac{f}{2}, \frac{e}{2})$ and $M_2 = (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\mathrm{F}}){N_1}^{\mathrm{T}})^{\smile} \cdot M_1 \cdot (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\mathrm{F}})N_1)^{\smile}$.
Then $(\mathbb{R} \to \mathbb{R}_{\mathrm{F}})M_2$ is symmetric.
Proof: $((\mathbb{R} \to \mathbb{R}_{\mathrm{F}}){N_1}^{\mathrm{T}})^{\mathrm{T}} = (\mathbb{R} \to \mathbb{R}_{\mathrm{F}})N_1$ by [3, (16)]. $(\mathbb{R} \to \mathbb{R}_{\mathrm{F}})M_2$ is symmetric by [3, (16)], (12), (7). □

(15) Let us consider real numbers $a_1$, $a_2$, $a_3$, $a_4$, $a_5$, $a_6$, $b_1$, $b_2$, $b_3$, $b_4$, $b_5$, $b_6$. Suppose symmetric3$(a_1, a_2, a_3, a_4, a_5, a_6) =$ symmetric3$(b_1, b_2, b_3, b_4, b_5, b_6)$. Then

(i) $a_1 = b_1$, and

(ii) $a_2 = b_2$, and

(iii) $a_3 = b_3$, and

(iv) $a_4 = b_4$, and

(v) $a_5 = b_5$, and

(vi) $a_6 = b_6$.

The theorem is a consequence of (2).

(16) Let us consider real numbers $a$, $b$, $c$, $d$, $e$, $f$, a point $P$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and an invertible square matrix $N$ over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose it is not true that $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and $e = 0$ and $f = 0$. Suppose that $P \in$ conic$(a, b, c, d, e, f)$. Let us consider real numbers $f_5$, $f_{12}$, $f_{19}$, $f_{20}$, $f_{21}$, $f_{23}$, $f_{22}$, square matrices $M_1$, $M_2$ over $\mathbb{R}$ of dimension 3, and a square matrix $N_1$ over $\mathbb{R}$ of dimension 3. Suppose $M_1 =$ symmetric3$(a, b, c, \frac{d}{2}, \frac{e}{2}, \frac{f}{2})$ and $N_1 = (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})N$ and $M_2 = (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\mathrm{F}}){N_1}^{\mathrm{T}})^{\smile} \cdot M_1 \cdot (\mathbb{R}_{\mathrm{F}} \to \mathbb{R})((\mathbb{R} \to \mathbb{R}_{\mathrm{F}})N_1)^{\smile}$ and $M_2 =$ symmetric3$(f_5, f_{21}, f_{23}, f_{12}, f_{19}, f_{22})$. Then

(i) it is not true that $f_5 = 0$ and $f_{21} = 0$ and $f_{23} = 0$ and $f_{12} = 0$ and $f_{22} = 0$ and $f_{19} = 0$, and

(ii) (the homography of $N)(P) \in$ conic$(f_5, f_{21}, f_{23}, 2 \cdot f_{12}, 2 \cdot f_{19}, 2 \cdot f_{22})$.

Proof: Consider $Q$ being a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P = Q$ and for every element $u$ of $\mathcal{E}^3_{\mathrm{T}}$ such that $u$ is not zero and $Q=$ the direction of $u$ holds $\mathrm{qfconic}(a,b,c,d,e,f,u)=0$. Reconsider $M=\mathrm{symmetric3}(a,b,c,\frac{d}{2},\frac{e}{2},\frac{f}{2})$ as a square matrix over $\mathbb{R}$ of dimension 3. Consider $u_{19}$, $v_3$ being elements of $\mathcal{E}^3_{\mathrm{T}}$, $u_{17}$ being a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$, $p_{11}$ being a finite sequence of elements of $\mathbb{R}^1$ such that $P=$ the direction of $u_{19}$ and $u_{19}$ is not zero and $u_{19}=u_{17}$ and $p_{11}=N\cdot u_{17}$ and $v_3=\mathrm{M2F}(p_{11})$ and $v_3$ is not zero and (the homography of $N$)$(P)=$ the direction of $v_3$. Reconsider $p_{10}=u_{19}$ as a finite sequence of elements of $\mathbb{R}$. $\mathrm{SumAll\,QuadraticForm}(p_{10},M,p_{10})=\mathrm{qfconic}(a,b,c,2\cdot\frac{d}{2},2\cdot\frac{e}{2},2\cdot\frac{f}{2},u_{19})$. Consider $a_8$, $b_8$, $c_{11}$, $d_4$, $e_5$, $f_{24}$, $g_2$, $h_2$, $i_2$ being elements of $\mathbb{R}_{\mathrm{F}}$ such that $N=\langle\langle a_8,b_8,c_{11}\rangle,\langle d_4,e_5,f_{24}\rangle,\langle g_2,h_2,i_2\rangle\rangle$. Reconsider $u_{10}=u_{17}$ as a finite sequence of elements of $\mathbb{R}$. Reconsider $N_1=(\mathbb{R}_{\mathrm{F}}\to\mathbb{R})N$ as a square matrix over $\mathbb{R}$ of dimension 3. Reconsider $M_2=(\mathbb{R}_{\mathrm{F}}\to\mathbb{R})((\mathbb{R}\to\mathbb{R}_{\mathrm{F}})N_1{}^{\mathrm{T}})^{\smile}\cdot M\cdot(\mathbb{R}_{\mathrm{F}}\to\mathbb{R})((\mathbb{R}\to\mathbb{R}_{\mathrm{F}})N_1)^{\smile}$ as a square matrix over $\mathbb{R}$ of dimension 3. $((\mathbb{R}\to\mathbb{R}_{\mathrm{F}})N_1{}^{\mathrm{T}})^{\mathrm{T}}=(\mathbb{R}\to\mathbb{R}_{\mathrm{F}})N_1$ by [3, (16)]. $(\mathbb{R}\to\mathbb{R}_{\mathrm{F}})M_2$ is symmetric by [3, (16)], (12), (7). Consider $m_1$, $m_2$, $m_3$, $m_4$, $m_5$, $m_6$, $m_7$, $m_8$, $m_9$ being elements of $\mathbb{R}_{\mathrm{F}}$ such that $M_2=\langle\langle m_1,m_2,m_3\rangle,\langle m_4,m_5,m_6\rangle,\langle m_7,m_8,m_9\rangle\rangle$. $m_2=m_4$ and $m_3=m_7$ and $m_8=m_6$. Reconsider $u_3=N_1\cdot u_{10}$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. $u_3$ is not zero by [5, (24)], [14, (59), (86)]. Reconsider $u_2=N_1\cdot u_{10}$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $f_5=m_1$, $f_{12}=m_2$, $f_{19}=m_3$, $f_{21}=m_5$, $f_{22}=m_6$, $f_{23}=m_9$ as a real number. $\mathrm{qfconic}(f_5,f_{21},f_{23},2\cdot f_{12},2\cdot f_{19},2\cdot f_{22},u_2)=0$. It is not true that $f_5=0$ and $f_{21}=0$ and $f_{23}=0$ and $2\cdot f_{12}=0$ and $2\cdot f_{22}=0$ and $2\cdot f_{19}=0$. $u_2=v_3$. For every real numbers $u_{11}$, $u_{12}$, $u_{13}$, $u_{14}$, $u_{15}$, $u_{18}$, $u_{16}$ and for every square matrices $U_1$, $U_2$ over $\mathbb{R}$ of dimension 3 and for every square matrix $U_3$ over $\mathbb{R}$ of dimension 3 such that $U_1=\mathrm{symmetric3}(a,b,c,\frac{d}{2},\frac{e}{2},\frac{f}{2})$ and $U_3=(\mathbb{R}_{\mathrm{F}}\to\mathbb{R})N$ and $U_2=(\mathbb{R}_{\mathrm{F}}\to\mathbb{R})((\mathbb{R}\to\mathbb{R}_{\mathrm{F}})U_3{}^{\mathrm{T}})^{\smile}\cdot U_1\cdot(\mathbb{R}_{\mathrm{F}}\to\mathbb{R})((\mathbb{R}\to\mathbb{R}_{\mathrm{F}})U_3)^{\smile}$ and $U_2=\mathrm{symmetric3}(u_{11},u_{15},u_{18},u_{12},u_{13},u_{16})$ holds it is not true that $u_{11}=0$ and $u_{15}=0$ and $u_{18}=0$ and $u_{12}=0$ and $u_{16}=0$ and $u_{13}=0$. (the homography of $N$)$(P)\in\mathrm{conic}(u_{11},u_{15},u_{18},2\cdot u_{12},2\cdot u_{13},2\cdot u_{16})$. □

(17) Let us consider real numbers $a$, $b$, $c$, $d$, $e$, $f$, points $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$ of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and an invertible square matrix $N$ over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose it is not true that $a=0$ and $b=0$ and $c=0$ and $d=0$ and $e=0$ and $f=0$. Suppose that $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6\in\mathrm{conic}(a,b,c,d,e,f)$. Then there exist real numbers $a_2$, $b_2$, $c_2$, $d_2$, $e_2$, $f_2$ such that

(i) it is not true that $a_2=0$ and $b_2=0$ and $c_2=0$ and $d_2=0$ and $e_2=0$ and $f_2=0$, and

(ii) (the homography of $N$)$(P_1)$, (the homography of $N$)$(P_2)$, (the homography of $N$)($P_3$), (the homography of $N$)($P_4$), (the homography of $N$)($P_5$), (the homography of $N$)($P_6$) $\in$ conic($a_2, b_2, c_2, d_2, e_2, f_2$).

The theorem is a consequence of (3), (14), (6), and (16).

From now on $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$, $i$ denote elements of $\mathbb{R}_{\rm F}$.

Now we state the proposition:

(18) (i) if qfconic$(a, b, c, d, e, f, [1, 0, 0]) = 0$, then $a = 0$, and

(ii) if qfconic$(a, b, c, d, e, f, [0, 1, 0]) = 0$, then $b = 0$, and

(iii) if qfconic$(a, b, c, d, e, f, [0, 0, 1]) = 0$, then $c = 0$, and

(iv) if qfconic$(0, 0, 0, d, e, f, [1, 1, 1]) = 0$, then $d + e + f = 0$.

## 3. Pascal's Theorem

In the sequel $M$ denotes a square matrix over $\mathbb{R}_{\rm F}$ of dimension 3, $e_1$, $e_2$, $e_3$, $f_1$, $f_2$, $f_3$ denote elements of $\mathbb{R}_{\rm F}$, $M_8$, $M_{14}$, $M_{20}$, $M_{21}$, $M_{22}$, $M_{19}$, $M_{13}$, $M_{10}$, $M_9$, $M_{12}$, $M_{16}$, $M_{17}$, $M_{11}$, $M_{15}$, $M_{18}$ denote square matrices over $\mathbb{R}_{\rm F}$ of dimension 3, and $r_1$, $r_2$ denote real numbers.

Now we state the proposition:

(19) Suppose $M_9 = \langle\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle e_1, e_2, e_3\rangle\rangle$ and $M_{12} = \langle\langle 1, 0, 0\rangle, \langle 0, 0, 1\rangle, \langle f_1, f_2, f_3\rangle\rangle$ and $M_{16} = \langle\langle 0, 1, 0\rangle, \langle 1, 1, 1\rangle, \langle f_1, f_2, f_3\rangle\rangle$ and $M_{17} = \langle\langle 0, 0, 1\rangle, \langle 1, 1, 1\rangle, \langle e_1, e_2, e_3\rangle\rangle$ and $M_{10} = \langle\langle 1, 0, 0\rangle, \langle 0, 1, 0\rangle, \langle f_1, f_2, f_3\rangle\rangle$ and $M_{11} = \langle\langle 1, 0, 0\rangle, \langle 0, 0, 1\rangle, \langle e_1, e_2, e_3\rangle\rangle$ and $M_{15} = \langle\langle 0, 1, 0\rangle, \langle 1, 1, 1\rangle, \langle e_1, e_2, e_3\rangle\rangle$ and $M_{18} = \langle\langle 0, 0, 1\rangle, \langle 1, 1, 1\rangle, \langle f_1, f_2, f_3\rangle\rangle$ and ($r_1 \neq 0$ or $r_2 \neq 0$) and $r_1 \cdot e_1 \cdot e_2 + (r_2 \cdot e_1 \cdot e_3) = r_1 + r_2 \cdot e_2 \cdot e_3$ and $r_1 \cdot f_1 \cdot f_2 + (r_2 \cdot f_1 \cdot f_3) = r_1 + r_2 \cdot f_2 \cdot f_3$. Then $\mathrm{Det}\, M_9 \cdot \mathrm{Det}\, M_{12} \cdot \mathrm{Det}\, M_{16} \cdot \mathrm{Det}\, M_{17} = \mathrm{Det}\, M_{10} \cdot \mathrm{Det}\, M_{11} \cdot \mathrm{Det}\, M_{15} \cdot \mathrm{Det}\, M_{18}$.

In the sequel $p_1$, $p_2$, $p_3$, $p_4$, $p_5$, $p_6$ denote points of $\mathcal{E}^3_{\rm T}$.

(20) Suppose $M_9 = \langle p_1, p_2, p_5\rangle$ and $M_{12} = \langle p_1, p_3, p_6\rangle$ and $M_{16} = \langle p_2, p_4, p_6\rangle$ and $M_{17} = \langle p_3, p_4, p_5\rangle$ and $M_{10} = \langle p_1, p_2, p_6\rangle$ and $M_{11} = \langle p_1, p_3, p_5\rangle$ and $M_{15} = \langle p_2, p_4, p_5\rangle$ and $M_{18} = \langle p_3, p_4, p_6\rangle$. Then

(i) $\mathrm{Det}\, M_9 = \langle|p_1, p_2, p_5|\rangle$, and

(ii) $\mathrm{Det}\, M_{12} = \langle|p_1, p_3, p_6|\rangle$, and

(iii) $\mathrm{Det}\, M_{16} = \langle|p_2, p_4, p_6|\rangle$, and

(iv) $\mathrm{Det}\, M_{17} = \langle|p_3, p_4, p_5|\rangle$, and

(v) $\mathrm{Det}\, M_{10} = \langle|p_1, p_2, p_6|\rangle$, and

(vi) $\mathrm{Det}\, M_{11} = \langle|p_1, p_3, p_5|\rangle$, and

(vii) $\mathrm{Det}\, M_{15} = \langle|p_2, p_4, p_5|\rangle$, and

(viii) Det $M_{18} = \langle|p_3, p_4, p_6|\rangle$.

From now on $p_7$, $p_8$, $p_9$ denote points of $\mathcal{E}^3_{\mathrm{T}}$.

(21) Suppose $\langle|p_1, p_5, p_9|\rangle = 0$. Then $\langle|p_1, p_5, p_7|\rangle \cdot \langle|p_2, p_5, p_9|\rangle = -(\langle|p_1, p_2, p_5|\rangle \cdot \langle|p_5, p_9, p_7|\rangle)$. The theorem is a consequence of (1).

(22) Suppose $\langle|p_1, p_6, p_8|\rangle = 0$. Then $\langle|p_1, p_2, p_6|\rangle \cdot \langle|p_3, p_6, p_8|\rangle = \langle|p_1, p_3, p_6|\rangle \cdot \langle|p_2, p_6, p_8|\rangle$. The theorem is a consequence of (1).

(23) Suppose $\langle|p_2, p_4, p_9|\rangle = 0$. Then $\langle|p_2, p_4, p_5|\rangle \cdot \langle|p_2, p_9, p_7|\rangle = -(\langle|p_2, p_4, p_7|\rangle \cdot \langle|p_2, p_5, p_9|\rangle)$.

(24) Suppose $\langle|p_2, p_6, p_7|\rangle = 0$. Then $\langle|p_2, p_4, p_7|\rangle \cdot \langle|p_2, p_6, p_8|\rangle = -(\langle|p_2, p_4, p_6|\rangle \cdot \langle|p_2, p_8, p_7|\rangle)$.

(25) Suppose $\langle|p_3, p_4, p_8|\rangle = 0$. Then $\langle|p_3, p_4, p_6|\rangle \cdot \langle|p_3, p_5, p_8|\rangle = \langle|p_3, p_4, p_5|\rangle \cdot \langle|p_3, p_6, p_8|\rangle$.

(26) Suppose $\langle|p_3, p_5, p_7|\rangle = 0$. Then $\langle|p_1, p_3, p_5|\rangle \cdot \langle|p_5, p_8, p_7|\rangle = -(\langle|p_1, p_5, p_7|\rangle \cdot \langle|p_3, p_5, p_8|\rangle)$. The theorem is a consequence of (1).

(27) Let us consider non zero real numbers $r_{125}$, $r_{136}$, $r_{246}$, $r_{345}$, $r_{126}$, $r_{135}$, $r_{245}$, $r_{346}$, $r_{157}$, $r_{259}$, $r_{597}$, $r_{368}$, $r_{268}$, $r_{297}$, $r_{247}$, $r_{287}$, $r_{358}$, $r_{587}$. Suppose $r_{125} \cdot r_{136} \cdot r_{246} \cdot r_{345} = r_{126} \cdot r_{135} \cdot r_{245} \cdot r_{346}$ and $r_{157} \cdot r_{259} = -(r_{125} \cdot r_{597})$ and $r_{126} \cdot r_{368} = r_{136} \cdot r_{268}$ and $r_{245} \cdot r_{297} = -(r_{247} \cdot r_{259})$ and $r_{247} \cdot r_{268} = -(r_{246} \cdot r_{287})$ and $r_{346} \cdot r_{358} = r_{345} \cdot r_{368}$ and $r_{135} \cdot r_{587} = -(r_{157} \cdot r_{358})$. Then $r_{287} \cdot r_{597} = r_{297} \cdot r_{587}$.

(28) Suppose $p_1 = \langle 1, 0, 0\rangle$ and $p_2 = \langle 0, 1, 0\rangle$ and $p_3 = \langle 0, 0, 1\rangle$ and $p_4 = \langle 1, 1, 1\rangle$ and $p_5 = \langle e_1, e_2, e_3\rangle$ and $p_6 = \langle f_1, f_2, f_3\rangle$ and qfconic$(0, 0, 0, r_1, r_2, -(r_1 + r_2), p_5) = 0$ and qfconic$(0, 0, 0, r_1, r_2, -(r_1 + r_2), p_6) = 0$. Then

(i) qfconic$(0, 0, 0, r_1, r_2, -(r_1 + r_2), p_1) = 0$, and

(ii) qfconic$(0, 0, 0, r_1, r_2, -(r_1 + r_2), p_2) = 0$, and

(iii) qfconic$(0, 0, 0, r_1, r_2, -(r_1 + r_2), p_3) = 0$, and

(iv) qfconic$(0, 0, 0, r_1, r_2, -(r_1 + r_2), p_4) = 0$, and

(v) $r_1 \cdot e_1 \cdot e_2 + (r_2 \cdot e_1 \cdot e_3) = r_1 + r_2 \cdot e_2 \cdot e_3$, and

(vi) $r_1 \cdot f_1 \cdot f_2 + (r_2 \cdot f_1 \cdot f_3) = r_1 + r_2 \cdot f_2 \cdot f_3$.

(29) Suppose $p_1 = \langle 1, 0, 0\rangle$ and $p_2 = \langle 0, 1, 0\rangle$ and $p_3 = \langle 0, 0, 1\rangle$ and $p_4 = \langle 1, 1, 1\rangle$ and $p_5 = \langle e_1, e_2, e_3\rangle$ and $p_6 = \langle f_1, f_2, f_3\rangle$ and $\langle|p_1, p_2, p_5|\rangle \neq 0$ and $\langle|p_1, p_3, p_6|\rangle \neq 0$ and $\langle|p_2, p_4, p_6|\rangle \neq 0$ and $\langle|p_3, p_4, p_5|\rangle \neq 0$ and $\langle|p_1, p_2, p_6|\rangle \neq 0$ and $\langle|p_1, p_3, p_5|\rangle \neq 0$ and $\langle|p_2, p_4, p_5|\rangle \neq 0$ and $\langle|p_3, p_4, p_6|\rangle \neq 0$ and $\langle|p_1, p_5, p_7|\rangle \neq 0$ and $\langle|p_2, p_5, p_9|\rangle \neq 0$ and $\langle|p_5, p_9, p_7|\rangle \neq 0$ and $\langle|p_3, p_6, p_8|\rangle \neq 0$ and $\langle|p_2, p_6, p_8|\rangle \neq 0$ and $\langle|p_2, p_9, p_7|\rangle \neq 0$ and $\langle|p_2, p_4, p_7|\rangle \neq 0$ and $\langle|p_2, p_8, p_7|\rangle \neq 0$ and $\langle|p_3, p_5, p_8|\rangle \neq 0$ and $\langle|p_5, p_8, p_7|\rangle$

$\neq 0$ and ($r_1 \neq 0$ or $r_2 \neq 0$) and $\mathrm{qfconic}(0,0,0,r_1,r_2,-(r_1+r_2),p_5) = 0$ and $\mathrm{qfconic}(0,0,0,r_1,r_2,-(r_1+r_2),p_6) = 0$ and $\langle|p_1,p_5,p_9|\rangle = 0$ and $\langle|p_1,p_6,p_8|\rangle = 0$ and $\langle|p_2,p_4,p_9|\rangle = 0$ and $\langle|p_2,p_6,p_7|\rangle = 0$ and $\langle|p_3,p_4,p_8|\rangle = 0$ and $\langle|p_3,p_5,p_7|\rangle = 0$. Then $\langle|p_2,p_8,p_7|\rangle \cdot \langle|p_5,p_9,p_7|\rangle = \langle|p_2,p_9,p_7|\rangle \cdot \langle|p_5,p_8,p_7|\rangle$. The theorem is a consequence of (20), (28), (19), (21), (22), (23), (24), (25), (26), and (27).

(30) Suppose $\langle|p_2,p_8,p_7|\rangle \cdot \langle|p_5,p_9,p_7|\rangle = \langle|p_2,p_9,p_7|\rangle \cdot \langle|p_5,p_8,p_7|\rangle$. Then $\langle|p_7,p_2,p_5|\rangle \cdot \langle|p_7,p_8,p_9|\rangle = 0$. The theorem is a consequence of (1).

(31) Let us consider a projective space $P_{10}$ defined in terms of collinearity, and elements $c_1$, $c_2$, $c_3$, $c_4$, $c_5$, $c_6$, $c_7$, $c_8$, $c_9$ of $P_{10}$. Suppose $c_1$, $c_2$ and $c_4$ are not collinear and $c_1$, $c_2$ and $c_5$ are not collinear and $c_1$, $c_6$ and $c_4$ are not collinear and $c_1$, $c_6$ and $c_5$ are not collinear and $c_2$, $c_6$ and $c_4$ are not collinear and $c_3$, $c_4$ and $c_2$ are not collinear and $c_3$, $c_4$ and $c_6$ are not collinear and $c_3$, $c_5$ and $c_2$ are not collinear and $c_3$, $c_5$ and $c_6$ are not collinear and $c_4$, $c_5$ and $c_2$ are not collinear and $c_1$, $c_4$ and $c_7$ are collinear and $c_1$, $c_5$ and $c_8$ are collinear and $c_2$, $c_3$ and $c_7$ are collinear and $c_2$, $c_5$ and $c_9$ are collinear and $c_6$, $c_3$ and $c_8$ are collinear and $c_6$, $c_4$ and $c_9$ are collinear. Then

(i) $c_9$, $c_2$ and $c_4$ are not collinear, and

(ii) $c_1$, $c_4$ and $c_9$ are not collinear, and

(iii) $c_2$, $c_3$ and $c_9$ are not collinear, and

(iv) $c_2$, $c_4$ and $c_7$ are not collinear, and

(v) $c_2$, $c_5$ and $c_8$ are not collinear, and

(vi) $c_2$, $c_9$ and $c_8$ are not collinear, and

(vii) $c_2$, $c_9$ and $c_7$ are not collinear, and

(viii) $c_6$, $c_4$ and $c_8$ are not collinear, and

(ix) $c_6$, $c_5$ and $c_8$ are not collinear, and

(x) $c_4$, $c_9$ and $c_8$ are not collinear, and

(xi) $c_4$, $c_9$ and $c_7$ are not collinear.

PROOF: For every elements $v_{102}$, $v_{103}$, $v_{100}$, $v_{104}$ of $P_{10}$, $v_{100} = v_{104}$ or $v_{104}$, $v_{100}$ and $v_{102}$ are not collinear or $v_{104}$, $v_{100}$ and $v_{103}$ are not collinear or $v_{102}$, $v_{103}$ and $v_{104}$ are collinear by [13, (5), (3)]. For every elements $v_{102}$, $v_{104}$, $v_{100}$, $v_{103}$ of $P_{10}$, $v_{100} = v_{103}$ or $v_{103}$, $v_{100}$ and $v_{102}$ are not collinear or $v_{103}$, $v_{100}$ and $v_{104}$ are not collinear or $v_{102}$, $v_{103}$ and $v_{104}$ are collinear by [13, (5), (3)]. For every elements $v_{102}$, $v_{103}$, $v_{104}$, $v_{101}$ of $P_{10}$, $v_{104} = v_{101}$ or $v_{101}$, $v_{104}$ and $v_{102}$ are not collinear or $v_{101}$, $v_{104}$ and $v_{103}$ are not collinear or $v_{102}$, $v_{103}$ and $v_{104}$ are collinear by [13, (2), (3)]. For every elements $v_{103}$,

$v_{104}$, $v_{102}$, $v_{101}$ of $P_{10}$, $v_{102} = v_{101}$ or $v_{101}$, $v_{102}$ and $v_{103}$ are not collinear or $v_{101}$, $v_{102}$ and $v_{104}$ are not collinear or $v_{102}$, $v_{103}$ and $v_{104}$ are collinear by [13, (2), (3)]. For every elements $v_2$, $v_{101}$, $v_{100}$ of $P_{10}$, $v_{101} = v_{100}$ or $v_{100}$, $v_{101}$ and $v_2$ are not collinear or $v_2$, $v_{101}$ and $v_{100}$ are collinear by [13, (2)]. □

In the sequel $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$, $P_7$, $P_8$, $P_9$ denote points of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ and $a$, $b$, $c$, $d$, $e$, $f$ denote real numbers.

Let $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$, $P_7$, $P_8$, $P_9$ be points of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. We say that $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$, $P_7$, $P_8$, $P_9$ form the Pascal configuration if and only if

(Def. 4) $P_1$, $P_2$ and $P_4$ are not collinear and $P_1$, $P_3$ and $P_4$ are not collinear and $P_2$, $P_3$ and $P_4$ are not collinear and $P_1$, $P_2$ and $P_5$ are not collinear and $P_1$, $P_2$ and $P_6$ are not collinear and $P_1$, $P_3$ and $P_5$ are not collinear and $P_1$, $P_3$ and $P_6$ are not collinear and $P_2$, $P_4$ and $P_5$ are not collinear and $P_2$, $P_4$ and $P_6$ are not collinear and $P_3$, $P_4$ and $P_5$ are not collinear and $P_3$, $P_4$ and $P_6$ are not collinear and $P_2$, $P_3$ and $P_5$ are not collinear and $P_2$, $P_3$ and $P_6$ are not collinear and $P_4$, $P_5$ and $P_1$ are not collinear and $P_4$, $P_6$ and $P_1$ are not collinear and $P_5$, $P_6$ and $P_1$ are not collinear and $P_5$, $P_6$ and $P_2$ are not collinear and $P_1$, $P_5$ and $P_9$ are collinear and $P_1$, $P_6$ and $P_8$ are collinear and $P_2$, $P_4$ and $P_9$ are collinear and $P_2$, $P_6$ and $P_7$ are collinear and $P_3$, $P_4$ and $P_8$ are collinear and $P_3$, $P_5$ and $P_7$ are collinear.

Now we state the propositions:

(32) Suppose $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$, $P_7$, $P_8$, $P_9$ form the Pascal configuration. Then

(i) $P_7$, $P_2$ and $P_5$ are not collinear, and

(ii) $P_1$, $P_5$ and $P_7$ are not collinear, and

(iii) $P_2$, $P_4$ and $P_7$ are not collinear, and

(iv) $P_2$, $P_5$ and $P_9$ are not collinear, and

(v) $P_2$, $P_6$ and $P_8$ are not collinear, and

(vi) $P_2$, $P_7$ and $P_8$ are not collinear, and

(vii) $P_2$, $P_7$ and $P_9$ are not collinear, and

(viii) $P_3$, $P_5$ and $P_8$ are not collinear, and

(ix) $P_3$, $P_6$ and $P_8$ are not collinear, and

(x) $P_5$, $P_7$ and $P_8$ are not collinear, and

(xi) $P_5$, $P_7$ and $P_9$ are not collinear.

The theorem is a consequence of (31).

(33) Suppose it is not true that $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and $e = 0$ and $f = 0$. Suppose that $\{P_1, P_2, P_3, P_4, P_5, P_6\} \subseteq \mathrm{conic}(a, b, c, d, e, f)$ and $P_1$, $P_2$ and $P_3$ are not collinear and $P_1$, $P_2$ and $P_4$ are not collinear and $P_1$, $P_3$ and $P_4$ are not collinear and $P_2$, $P_3$ and $P_4$ are not collinear and $P_7$, $P_2$ and $P_5$ are not collinear and $P_1$, $P_2$ and $P_5$ are not collinear and $P_1$, $P_2$ and $P_6$ are not collinear and $P_1$, $P_3$ and $P_5$ are not collinear and $P_1$, $P_3$ and $P_6$ are not collinear and $P_1$, $P_5$ and $P_7$ are not collinear and $P_2$, $P_4$ and $P_5$ are not collinear and $P_2$, $P_4$ and $P_6$ are not collinear and $P_2$, $P_4$ and $P_7$ are not collinear and $P_2$, $P_5$ and $P_9$ are not collinear and $P_2$, $P_6$ and $P_8$ are not collinear and $P_2$, $P_7$ and $P_8$ are not collinear and $P_2$, $P_7$ and $P_9$ are not collinear and $P_3$, $P_4$ and $P_5$ are not collinear and $P_3$, $P_4$ and $P_6$ are not collinear and $P_3$, $P_5$ and $P_8$ are not collinear and $P_3$, $P_6$ and $P_8$ are not collinear and $P_5$, $P_7$ and $P_8$ are not collinear and $P_5$, $P_7$ and $P_9$ are not collinear and $P_1$, $P_5$ and $P_9$ are collinear and $P_1$, $P_6$ and $P_8$ are collinear and $P_2$, $P_4$ and $P_9$ are collinear and $P_2$, $P_6$ and $P_7$ are collinear and $P_3$, $P_4$ and $P_8$ are collinear and $P_3$, $P_5$ and $P_7$ are collinear. Then $P_7$, $P_8$ and $P_9$ are collinear.

PROOF: Consider $N$ being an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that (the homography of $N)(P_1) = \mathrm{Dir100}$ and (the homography of $N)(P_2) = \mathrm{Dir010}$ and (the homography of $N)(P_3) = \mathrm{Dir001}$ and (the homography of $N)(P_4) = \mathrm{Dir111}$. Consider $u_5$ being a point of $\mathcal{E}^3_{\mathrm{T}}$ such that $u_5$ is not zero and (the homography of $N)(P_5) =$ the direction of $u_5$. Reconsider $p_{51} = u_5(1)$, $p_{52} = u_5(2)$, $p_{53} = u_5(3)$ as a real number. Consider $u_6$ being a point of $\mathcal{E}^3_{\mathrm{T}}$ such that $u_6$ is not zero and (the homography of $N)(P_6) =$ the direction of $u_6$. Reconsider $p_{61} = u_6(1)$, $p_{62} = u_6(2)$, $p_{63} = u_6(3)$ as a real number. Consider $u_7$ being a point of $\mathcal{E}^3_{\mathrm{T}}$ such that $u_7$ is not zero and (the homography of $N)(P_7) =$ the direction of $u_7$. Reconsider $p_{71} = u_7(1)$, $p_{72} = u_7(2)$, $p_{73} = u_7(3)$ as a real number. Consider $u_8$ being a point of $\mathcal{E}^3_{\mathrm{T}}$ such that $u_8$ is not zero and (the homography of $N)(P_8) =$ the direction of $u_8$. Reconsider $p_{81} = u_8(1)$, $p_{82} = u_8(2)$, $p_{83} = u_8(3)$ as a real number. Consider $u_9$ being a point of $\mathcal{E}^3_{\mathrm{T}}$ such that $u_9$ is not zero and (the homography of $N)(P_9) =$ the direction of $u_9$. Reconsider $p_{91} = u_9(1)$, $p_{92} = u_9(2)$, $p_{93} = u_9(3)$ as a real number. Consider $a_2$, $b_2$, $c_2$, $d_2$, $e_2$, $f_2$ being real numbers such that it is not true that $a_2 = 0$ and $b_2 = 0$ and $c_2 = 0$ and $d_2 = 0$ and $e_2 = 0$ and $f_2 = 0$. (the homography of $N)(P_1) \in \mathrm{conic}(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of $N)(P_2) \in \mathrm{conic}(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of $N)(P_3) \in \mathrm{conic}(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of $N)(P_4) \in \mathrm{conic}(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of $N)(P_5) \in \mathrm{conic}(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of $N)(P_6) \in \mathrm{conic}(a_2, b_2, c_2, d_2, e_2, f_2)$. Consider $P$ being a point of the projective space over $\mathcal{E}^3_{\rm T}$ such that the direction of $[1,0,0] = P$ and for every element $u$ of $\mathcal{E}^3_{\rm T}$ such that $u$ is not zero and $P =$ the direction of $u$ holds qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, u) = 0$. qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [1,0,0]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [0,1,0]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [0,0,1]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [1,1,1]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [p_{51}, p_{52}, p_{53}]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [p_{61}, p_{62}, p_{63}]) = 0$ by [4, (10)], [8, (3)]. Reconsider $a_7 = a_2$, $b_7 = b_2$, $c_{10} = c_2$, $d_3 = d_2$, $e_4 = e_2$, $f_4 = f_2$ as an element of $\mathbb{R}_{\rm F}$. $a_7 = 0$ and $b_7 = 0$ and $c_{10} = 0$. $a_7 = 0$ and $b_7 = 0$ and $c_{10} = 0$ and $d_3 + e_4 + f_4 = 0$. Reconsider $p_2 = \langle 0,1,0 \rangle$, $p_5 = \langle p_{51}, p_{52}, p_{53} \rangle$, $p_7 = \langle p_{71}, p_{72}, p_{73} \rangle$, $p_8 = \langle p_{81}, p_{82}, p_{83} \rangle$, $p_9 = \langle p_{91}, p_{92}, p_{93} \rangle$ as a point of $\mathcal{E}^3_{\rm T}$. $\langle | p_7, p_2, p_5 | \rangle \neq 0$ by [3, (102)], [8, (3)], [3, (43)], [4, (10)]. $\langle | p_2, p_8, p_7 | \rangle \cdot \langle | p_5, p_9, p_7 | \rangle = \langle | p_2, p_9, p_7 | \rangle \cdot \langle | p_5, p_8, p_7 | \rangle$. $\langle | p_7, p_2, p_5 | \rangle \cdot \langle | p_7, p_8, p_9 | \rangle = 0$. □

(34) Suppose it is not true that $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and $e = 0$ and $f = 0$. Suppose that $\{P_1, P_2, P_3, P_4, P_5, P_6\} \subseteq$ conic$(a, b, c, d, e, f)$ and $P_1$, $P_2$ and $P_3$ are not collinear and $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$, $P_7$, $P_8$, $P_9$ form the Pascal configuration. Then $P_7$, $P_8$ and $P_9$ are collinear. The theorem is a consequence of (32) and (33).

Note that $\mathcal{E}^3_{\rm T}$ is up 3-dimensional.

(35) Suppose it is not true that $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and $e = 0$ and $f = 0$. Suppose that $\{P_1, P_2, P_3, P_4, P_5, P_6\} \subseteq$ conic$(a, b, c, d, e, f)$ and $P_1$, $P_2$ and $P_3$ are collinear and $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$, $P_7$, $P_8$, $P_9$ form the Pascal configuration. Then $P_7$, $P_8$ and $P_9$ are collinear.

PROOF: Consider $N$ being an invertible square matrix over $\mathbb{R}_{\rm F}$ of dimension 3 such that (the homography of $N)(P_1) =$ Dir100 and (the homography of $N)(P_2) =$ Dir010 and (the homography of $N)(P_4) =$ Dir001 and (the homography of $N)(P_5) =$ Dir111. Consider $u_3$ being a point of $\mathcal{E}^3_{\rm T}$ such that $u_3$ is not zero and (the homography of $N)(P_3) =$ the direction of $u_3$. Reconsider $p_{31} = u_3(1)$, $p_{32} = u_3(2)$, $p_{33} = u_3(3)$ as a real number. Consider $u_6$ being a point of $\mathcal{E}^3_{\rm T}$ such that $u_6$ is not zero and (the homography of $N)(P_6) =$ the direction of $u_6$. Reconsider $p_{61} = u_6(1)$, $p_{62} = u_6(2)$, $p_{63} = u_6(3)$ as a real number. Consider $a_2$, $b_2$, $c_2$, $d_2$, $e_2$, $f_2$ being real numbers such that it is not true that $a_2 = 0$ and $b_2 = 0$ and $c_2 = 0$ and $d_2 = 0$ and $e_2 = 0$ and $f_2 = 0$ and (the homography of $N)(P_1) \in$ conic$(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of $N)(P_2) \in$ conic$(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of $N)(P_3) \in$ conic$(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of $N)(P_4) \in$ conic$(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of $N)(P_5) \in$ conic$(a_2, b_2, c_2, d_2, e_2, f_2)$ and (the homography of $N)(P_6) \in$ conic$(a_2, b_2, c_2, d_2, e_2, f_2)$. Consider $P$ being a point of the projective space over $\mathcal{E}^3_{\rm T}$ such that the direction of $[1,0,0] = P$ and for every element $u$ of $\mathcal{E}^3_{\mathrm{T}}$ such that $u$ is not zero and $P =$ the direction of $u$ holds qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, u) = 0$. qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [1, 0, 0]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [0, 1, 0]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [0, 0, 1]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [1, 1, 1]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [p_{31}, p_{32}, p_{33}]) = 0$ and qfconic$(a_2, b_2, c_2, d_2, e_2, f_2, [p_{61}, p_{62}, p_{63}]) = 0$ by [4, (10)], [8, (3)]. Reconsider $a_7 = a_2$, $b_7 = b_2$, $c_{10} = c_2$, $d_3 = d_2$, $e_4 = e_2$, $f_4 = f_2$ as an element of $\mathbb{R}_{\mathrm{F}}$. $a_7 = 0$ and $b_7 = 0$ and $c_{10} = 0$. $a_7 = 0$ and $b_7 = 0$ and $c_{10} = 0$ and $d_3 + e_4 + f_4 = 0$. Reconsider $p_1 = \langle 1, 0, 0\rangle$, $p_2 = \langle 0, 1, 0\rangle$, $p_3 = \langle p_{31}, p_{32}, p_{33}\rangle$ as a point of $\mathcal{E}^3_{\mathrm{T}}$. $\langle|p_1, p_2, p_3|\rangle = 0$ by [3, (102)], [10, (23)], [9, (25)], [4, (10)]. $p_{31} \neq 0$ and $p_{32} \neq 0$ by [8, (2), (8), (4)]. □

(36) PASCAL'S THEOREM:
Suppose it is not true that $a = 0$ and $b = 0$ and $c = 0$ and $d = 0$ and $e = 0$ and $f = 0$. Suppose that $\{P_1, P_2, P_3, P_4, P_5, P_6\} \subseteq$ conic$(a, b, c, d, e, f)$ and $P_1$, $P_2$, $P_3$, $P_4$, $P_5$, $P_6$, $P_7$, $P_8$, $P_9$ form the Pascal configuration. Then $P_7$, $P_8$ and $P_9$ are collinear. The theorem is a consequence of (35) and (34).

## References


[1] Jesse Alama. Escape to Mizar for ATPs. *arXiv preprint arXiv:1204.6615*, 2012.

[2] Grzegorz Bancerek, Czesław Byliński, Adam Grabowski, Artur Korniłowicz, Roman Matuszewski, Adam Naumowicz, Karol Pąk, and Josef Urban. Mizar: State-of-the-art and beyond. In Manfred Kerber, Jacques Carette, Cezary Kaliszyk, Florian Rabe, and Volker Sorge, editors, *Intelligent Computer Mathematics*, volume 9150 of *Lecture Notes in Computer Science*, pages 261–279. Springer International Publishing, 2015. ISBN 978-3-319-20614-1. doi:10.1007/978-3-319-20615-8_17.

[3] Roland Coghetto. Homography in $\mathbb{RP}^2$. *Formalized Mathematics*, 24(**4**):239–251, 2016. doi:10.1515/forma-2016-0020.

[4] Roland Coghetto. Group of homography in real projective plane. *Formalized Mathematics*, 25(**1**):55–62, 2017. doi:10.1515/forma-2017-0005.

[5] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[6] Adam Grabowski. Solving two problems in general topology via types. In *Types for Proofs and Programs, International Workshop, TYPES 2004, Jouy-en-Josas, France, December 15-18, 2004, Revised Selected Papers*, pages 138–153, 2004. doi:10.1007/11617990_9.

[7] Adam Grabowski. Mechanizing complemented lattices within Mizar system. *Journal of Automated Reasoning*, 55:211–221, 2015. doi:10.1007/s10817-015-9333-5.

[8] Kanchun, Hiroshi Yamazaki, and Yatsuka Nakamura. Cross products and tripple vector products in 3-dimensional Euclidean space. *Formalized Mathematics*, 11(**4**):381–383, 2003.

[9] Wojciech Leończuk and Krzysztof Prażmowski. A construction of analytical projective space. *Formalized Mathematics*, 1(**4**):761–766, 1990.

[10] Wojciech Leończuk and Krzysztof Prażmowski. Projective spaces – part I. *Formalized Mathematics*, 1(**4**):767–776, 1990.

[11] Jürgen Richter-Gebert. *Pappos's Theorem: Nine Proofs and Three Variations*, pages 3–31. Springer Berlin Heidelberg, 2011. ISBN 978-3-642-17286-1. doi:10.1007/978-3-642-17286-1_1.

[12] Piotr Rudnicki and Josef Urban. Escape to ATP for Mizar. In *First International Workshop on Proof eXchange for Theorem Proving-PxTP 2011*, 2011.

[13] Wojciech Skaba. The collinearity structure. *Formalized Mathematics*, 1(**4**):657–659, 1990.

[14] Nobuyuki Tamura and Yatsuka Nakamura. Determinant and inverse of matrices of real elements. *Formalized Mathematics*, 15(**3**):127–136, 2007. doi:10.2478/v10037-007-0014-7.

*Received June 27, 2017*

---

The English version of this volume of Formalized Mathematics was financed under agreement 548/P-DUN/2016 with the funds from the Polish Minister of Science and Higher Education for the dissemination of science.