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About Quotient Orders and Ordering Sequences

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Summary. In preparation for the formalization in Mizar [4] of lotteries as given in [14], this article closes some gaps in the Mizar Mathematical Library (MML) regarding relational structures. The quotient order is introduced by the equivalence relation identifying two elements x, y of a preorder as equivalent if $x \leqslant y$ and $y \leqslant x$. This concept is known (see e.g. chapter 5 of [19]) and was first introduced into the MML in [13] and that work is incorporated here. Furthermore given a set A, partition D of A and a finite-support function $f : A \to \mathbb{R}$, a function $\Sigma_f : D \to \mathbb{R}, \Sigma_f(X) = \sum_{x \in X} f(x)$ can be defined as some kind of natural "restriction" from f to D. The first main result of this article can then be formulated as:

$$\sum_{x \in A} f(x) = \sum_{X \in D} \Sigma_f(X) \left(= \sum_{X \in D} \sum_{x \in X} f(x) \right)$$

After that (weakly) ascending/descending finite sequences (based on [3]) are introduced, in analogous notation to their infinite counterparts introduced in [18] and [13].

The second main result is that any finite subset of any transitive connected relational structure can be sorted as a ascending or descending finite sequence, thus generalizing the results from [16], where finite sequence of real numbers were sorted.

The third main result of the article is that any weakly ascending/weakly descending finite sequence on elements of a preorder induces a weakly ascending/weakly descending finite sequence on the projection of these elements into the quotient order. Furthermore, weakly ascending finite sequences can be interpreted as directed walks in a directed graph, when the set of edges is described by ordered pairs of vertices, which is quite common (see e.g. [10]).

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Additionally, some auxiliary theorems are provided, e.g. two schemes to find the smallest or the largest element in a finite subset of a connected transitive relational structure with a given property and a lemma I found rather useful: Given two finite one-to-one sequences s, t on a set X, such that $\operatorname{rng} t \subseteq \operatorname{rng} s$, and a function $f : X \to \mathbb{R}$ such that f is zero for every $x \in \operatorname{rng} s \setminus \operatorname{rng} t$, we have $\sum f \circ s = \sum f \circ t$.

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1. Preliminaries

Now we state the proposition:

(1) Let us consider sets A, B, and an object x. If $A = B \setminus \{x\}$ and $x \in B$, then $B \setminus A = \{x\}$.

Let Y be a set and X be a subset of Y. One can verify that every binary relation which is X-defined is also Y-defined.

Now we state the propositions:

(2) Let us consider a set X, and an object x. If $x \in X$ and $\overline{\overline{X}} = 1$, then $\{x\} = X$.

(3) Let us consider a set X, and a natural number k. Suppose $X \subseteq \operatorname{Seg} k$. Then $\operatorname{rng} \operatorname{Sgm} X \subseteq \operatorname{Seg} k$.

Let s be a finite sequence and N be a subset of $\operatorname{dom} s$. Observe that $s \cdot \operatorname{Sgm} N$ is finite sequence-like.

Let A be a set, B be a subset of A, C be a non empty set, f be a finite sequence of elements of B, and g be a function from A into C. Let us observe that $g \cdot f$ is finite sequence-like.

Let s be a finite sequence. Let us observe that $s \cdot \operatorname{idseq}(\operatorname{len} s)$ is finite sequence-like.

One can verify that $\operatorname{Rev}(\operatorname{Rev}(s))$ reduces to s.

Let X be a set. Note that there exists a subset of X which is finite.

The scheme *Finite2* deals with a set $\mathcal{A}$ and a subset $\mathcal{B}$ of $\mathcal{A}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 1) $\mathcal{P}[\mathcal{A}]$

provided

- $\mathcal{A}$ is finite and
- $\mathcal{P}[\mathcal{B}]$ and

- for every sets x, C such that $x \in \mathcal{A} \setminus \mathcal{B}$ and $\mathcal{B} \subseteq C \subseteq \mathcal{A}$ and $\mathcal{P}[C]$ holds $\mathcal{P}[C \cup \{x\}]$.

Let A be an empty set. One can check that every partition of A is empty and there exists a partition of A which is empty.

Let S, T be 1-sorted structures, f be a function from S into T, and B be a subset of S. Let us observe that the functor $f^\circ B$ yields a subset of T. Now we state the proposition:

(4) Let us consider a set X, an order R in X, a finite subset B of X, and an object x. If $B = \{x\}$, then $\mathrm{SgmX}(R, B) = \langle x \rangle$.
PROOF: Set $f = \langle x \rangle$. For every natural numbers n, m such that $n, m \in \operatorname{dom} f$ and $n < m$ holds $f_n \neq f_m$ and $\langle f_n, f_m \rangle \in R$ by [3, (38), (2)]. $\square$

Let us consider a finite sequence s of elements of $\mathbb{R}$. Now we state the propositions:

(5) If $\sum s \neq 0$, then there exists a natural number i such that $i \in \operatorname{dom} s$ and $s(i) \neq 0$.

(6) If s is non-negative yielding and there exists a natural number i such that $i \in \operatorname{dom} s$ and $s(i) \neq 0$, then $\sum s > 0$.
PROOF: Consider i being a natural number such that $i \in \operatorname{dom} s$ and $s(i) \neq 0$. Set $s_1 = s$. For every natural number j such that $j \in \operatorname{dom} s_1$ holds $0 \leqslant s_1(j)$ by [6, (3)]. There exists a natural number k such that $k \in \operatorname{dom} s_1$ and $0 < s_1(k)$ by [6, (3)]. $\square$

(7) If s is non-positive yielding and there exists a natural number i such that $i \in \operatorname{dom} s$ and $s(i) \neq 0$, then $\sum s < 0$.
PROOF: Reconsider $s_1 = -s$ as a finite sequence of elements of $\mathbb{R}$. There exists a natural number i such that $i \in \operatorname{dom} s_1$ and $s_1(i) \neq 0$ by [12, (58)]. $\sum s_1 > 0$. $\square$

(8) Let us consider a set X, finite sequences s, t of elements of X, and a function f from X into $\mathbb{R}$. Suppose s is one-to-one and t is one-to-one and $\operatorname{rng} t \subseteq \operatorname{rng} s$ and for every element x of X such that $x \in \operatorname{rng} s \setminus \operatorname{rng} t$ holds $f(x) = 0$. Then $\sum(f \cdot s) = \sum(f \cdot t)$.
PROOF: Define $\mathcal{P}[\text{set}] \equiv$ there exists a finite sequence r of elements of X such that r is one-to-one and $\operatorname{rng} t \subseteq \operatorname{rng} r$ and $\operatorname{rng} r = \$_1$ and $\sum(f \cdot r) = \sum(f \cdot t)$. Reconsider $r_1 = \operatorname{rng} t$ as a subset of $\operatorname{rng} s$. For every sets x, C such that $x \in \operatorname{rng} s \setminus r_1$ and $r_1 \subseteq C \subseteq \operatorname{rng} s$ and $\mathcal{P}[C]$ holds $\mathcal{P}[C \cup \{x\}]$ by [9, (40)], [3, (38), (31)], [9, (31)]. $\mathcal{P}[\operatorname{rng} s]$ from *Finite2*. Consider r being a finite sequence of elements of X such that r is one-to-one and $\operatorname{rng} t \subseteq \operatorname{rng} r$ and $\operatorname{rng} r = \operatorname{rng} s$ and $\sum(f \cdot r) = \sum(f \cdot t)$. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv r(\$_1) = s(\$_2)$. For every object i such that $i \in \operatorname{dom} r$ there exists an object j such that $j \in \operatorname{dom} s$ and $\mathcal{Q}[i, j]$ by [6, (3)]. Consider p being a function

from $\operatorname{dom} r$ into $\operatorname{dom} s$ such that for every object x such that $x \in \operatorname{dom} r$ holds $\mathcal{Q}[x, p(x)]$ from [7, Sch. 1]. p is a permutation of $\operatorname{dom} r$ by [21, (63)]. For every object i, $i \in \operatorname{dom} r$ iff $i \in \operatorname{dom} p$ and $p(i) \in \operatorname{dom} s$ by [6, (3)]. For every object x, $x \in \operatorname{dom}(f \cdot s)$ iff $x \in \operatorname{dom} s$ by [6, (11), (3)]. $\square$

Let X be a set, f be a function, and g be a positive yielding function from X into $\mathbb{R}$. Let us observe that $g \cdot f$ is positive yielding.

Let g be a negative yielding function from X into $\mathbb{R}$. Note that $g \cdot f$ is negative yielding.

Let g be a non-positive yielding function from X into $\mathbb{R}$. Let us observe that $g \cdot f$ is non-positive yielding.

Let g be a non-negative yielding function from X into $\mathbb{R}$. Note that $g \cdot f$ is non-negative yielding.

Let s be a function. Note that the functor support s yields a subset of $\operatorname{dom} s$. Let X be a set. Let us observe that there exists a function from X into $\mathbb{R}$ which is finite-support and non-negative yielding and there exists a function from X into $\mathbb{C}$ which is non-negative yielding and finite-support.

Now we state the proposition:

(9) Let us consider a set A, and a function f from A into $\mathbb{C}$. Then support $f =$ support$(-f)$.

PROOF: For every object x, $x \in$ support f iff $x \in$ support$(-f)$ by [15, (5), (66)]. $\square$

Let A be a set and f be a finite-support function from A into $\mathbb{C}$. Observe that $-f$ is finite-support.

Let f be a finite-support function from A into $\mathbb{R}$. One can verify that $-f$ is finite-support.

2. Orders

Let us consider a set X, a binary relation R, and a subset Y of X. Now we state the propositions:

(10) If R is irreflexive in X, then R is irreflexive in Y.

(11) If R is symmetric in X, then R is symmetric in Y.

(12) If R is asymmetric in X, then R is asymmetric in Y.

Let A be a relational structure. We say that A is connected if and only if

(Def. 1) the internal relation of A is connected in the carrier of A.

We say that A is strongly connected if and only if

(Def. 2) the internal relation of A is strongly connected in the carrier of A.

Let us note that there exists a relational structure which is non empty, reflexive, transitive, antisymmetric, connected, strongly connected, strict, and total and every relational structure which is strongly connected is also reflexive and connected and every relational structure which is reflexive and connected is also strongly connected and every relational structure which is empty is also reflexive, antisymmetric, transitive, connected, and strongly connected.

Let A be a relational structure and a_1, a_2 be elements of A. We say that $a_1 \approx a_2$ if and only if

(Def. 3) $a_1 \leqslant a_2 \leqslant a_1$.

Now we state the proposition:

(13) Let us consider a reflexive, non empty relational structure A, and an element a of A. Then $a \approx a$.

Let A be a reflexive, non empty relational structure and a_1, a_2 be elements of A. One can verify that the predicate $a_1 \approx a_2$ is reflexive.

Let A be a relational structure. We say that $a_1 \lnapprox a_2$ if and only if

(Def. 4) $a_1 \leqslant a_2$ and $a_2 \nleqslant a_1$.

Observe that the predicate is irreflexive.

We introduce the notation $a_2 \gnapprox a_1$ as a synonym of $a_1 \lnapprox a_2$.

Let A be a connected relational structure. One can verify that the predicate $a_1 \lnapprox a_2$ is asymmetric.

Now we state the propositions:

(14) Let us consider a non empty relational structure A, and elements a_1, a_2 of A. Suppose A is strongly connected. Then

(i) $a_1 \lnapprox a_2$, or

(ii) $a_1 \approx a_2$, or

(iii) $a_1 \gnapprox a_2$.

(15) Let us consider a transitive relational structure A, and elements a_1, a_2, a_3 of A. Then

(i) if $a_1 \lnapprox a_2$ and $a_2 \leqslant a_3$, then $a_1 \lnapprox a_3$, and

(ii) if $a_1 \leqslant a_2$ and $a_2 \lnapprox a_3$, then $a_1 \lnapprox a_3$.

(16) Let us consider a non empty relational structure A, and elements a_1, a_2 of A. If A is strongly connected, then $a_1 \leqslant a_2$ or $a_2 \leqslant a_1$.

(17) Let us consider a non empty relational structure A, a subset B of A, and elements a_1, a_2 of A. Suppose the internal relation of A is connected in B and a_1, $a_2 \in B$ and $a_1 \neq a_2$. Then

(i) $a_1 \leqslant a_2$, or

(ii) $a_2 \leqslant a_1$.

Let us consider a non empty relational structure A and elements a_1, a_2 of A. Now we state the propositions:

(18) If A is connected and $a_1 \neq a_2$, then $a_1 \leqslant a_2$ or $a_2 \leqslant a_1$.

(19) If A is strongly connected, then $a_1 = a_2$ or $a_1 < a_2$ or $a_2 < a_1$. The theorem is a consequence of (16).

Let us consider a relational structure A and elements a_1, a_2 of A. Now we state the propositions:

(20) If $a_1 \leqslant a_2$, then $a_1, a_2 \in$ the carrier of A.

(21) If $a_1 \leqslant a_2$, then A is not empty.

(22) Let us consider a transitive relational structure A, and a finite subset B of A. Suppose B is not empty and the internal relation of A is connected in B. Then there exists an element x of A such that

(i) $x \in B$, and

(ii) for every element y of A such that $y \in B$ and $x \neq y$ holds $x \leqslant y$.

Proof: Define $\mathcal{P}[\text{set}] \equiv$ if $\$_1$ is not empty, then there exists an element x of A such that $x \in \$_1$ and for every element y of A such that $y \in \$_1$ and $x \neq y$ holds $x \leqslant y$. For every sets z, C such that $z \in B$ and $C \subseteq B$ and $\mathcal{P}[C]$ holds $\mathcal{P}[C \cup \{z\}]$ by [9, (31)], (17), [9, (136)], [22, (3)]. $\mathcal{P}[B]$ from [11, Sch. 2]. $\square$

(23) Let us consider a connected, transitive relational structure A, and a finite subset B of A. Suppose B is not empty. Then there exists an element x of A such that

(i) $x \in B$, and

(ii) for every element y of A such that $y \in B$ and $x \neq y$ holds $x \leqslant y$.

The theorem is a consequence of (22).

(24) Let us consider a transitive relational structure A, and a finite subset B of A. Suppose B is not empty and the internal relation of A is connected in B. Then there exists an element x of A such that

(i) $x \in B$, and

(ii) for every element y of A such that $y \in B$ and $x \neq y$ holds $y \leqslant x$.

Proof: Define $\mathcal{P}[\text{set}] \equiv$ if $\$_1$ is not empty, then there exists an element x of A such that $x \in \$_1$ and for every element y of A such that $y \in \$_1$ and $x \neq y$ holds $y \leqslant x$. For every sets z, C such that $z \in B$ and $C \subseteq B$ and $\mathcal{P}[C]$ holds $\mathcal{P}[C \cup \{z\}]$ by [9, (31)], (17), [9, (136)], [22, (3)]. $\mathcal{P}[B]$ from [11, Sch. 2]. $\square$

(25) Let us consider a connected, transitive relational structure A, and a finite subset B of A. Suppose B is not empty. Then there exists an element x of A such that

(i) $x \in B$, and

(ii) for every element y of A such that $y \in B$ and $x \neq y$ holds $y \leqslant x$.

The theorem is a consequence of (24).

A preorder is a reflexive, transitive relational structure.

A linear preorder is a strongly connected, transitive relational structure.

An order is a reflexive, antisymmetric, transitive relational structure.

A linear order is a strongly connected, antisymmetric, transitive relational structure. Let us observe that every preorder is quasi-ordered and there exists a linear order which is empty.

Now we state the propositions:

(26) Let us consider a preorder A. Then the internal relation of A quasi-orders the carrier of A.

(27) Let us consider an order A. Then the internal relation of A partially orders the carrier of A.

(28) Let us consider a linear order A. Then the internal relation of A linearly orders the carrier of A.

Let us consider a relational structure A. Now we state the propositions:

(29) If the internal relation of A quasi-orders the carrier of A, then A is reflexive and transitive.

(30) If the internal relation of A partially orders the carrier of A, then A is reflexive, transitive, and antisymmetric.

(31) If the internal relation of A linearly orders the carrier of A, then A is reflexive, transitive, antisymmetric, and connected.

The scheme *RelStrMin* deals with a transitive, connected relational structure $\mathcal{A}$ and a finite subset $\mathcal{B}$ of $\mathcal{A}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 2) There exists an element x of $\mathcal{A}$ such that $x \in \mathcal{B}$ and $\mathcal{P}[x]$ and for every element y of $\mathcal{A}$ such that $y \in \mathcal{B}$ and $y \lnsim x$ holds $\mathcal{P}[y]$

provided

- there exists an element x of $\mathcal{A}$ such that $x \in \mathcal{B}$ and $\mathcal{P}[x]$.

The scheme *RelStrMax* deals with a transitive, connected relational structure $\mathcal{A}$ and a finite subset $\mathcal{B}$ of $\mathcal{A}$ and a unary predicate $\mathcal{P}$ and states that

(Sch. 3) There exists an element x of $\mathcal{A}$ such that $x \in \mathcal{B}$ and $\mathcal{P}[x]$ and for every element y of $\mathcal{A}$ such that $y \in \mathcal{B}$ and $x \lnsim y$ holds $\mathcal{P}[y]$

provided

- there exists an element x of $\mathcal{A}$ such that $x \in \mathcal{B}$ and $\mathcal{P}[x]$.

3. Quotient Order

Let A be a set and D be a partition of A. The functor EqRelOf(D) yielding an equivalence relation of A is defined by

(Def. 5) $D = \mathrm{Classes}\, it$.

Let A be a preorder. The functor EqRelOf(A) yielding an equivalence relation of the carrier of A is defined by

(Def. 6) for every elements x, y of A, $\langle x, y\rangle \in it$ iff $x \leqslant y \leqslant x$.

Now we state the proposition:

(32) Let us consider a preorder A. Then EqRelOf$(A) =$ EqRel(A).

Let A be an empty preorder. Let us note that EqRelOf(A) is empty.

Let A be a non empty preorder. Observe that EqRelOf(A) is non empty.

Now we state the proposition:

(33) Let us consider an order A. Then EqRelOf$(A) = \mathrm{id}_{\alpha}$, where α is the carrier of A.

Let A be a preorder. The functor QuotientOrder(A) yielding a strict relational structure is defined by

(Def. 7) the carrier of it = Classes(EqRelOf(A)) and for every elements X, Y of Classes(EqRelOf(A)), $\langle X, Y\rangle \in$ the internal relation of it iff there exist elements x, y of A such that $X = [x]_{\mathrm{EqRelOf}(A)}$ and $Y = [y]_{\mathrm{EqRelOf}(A)}$ and $x \leqslant y$.

Let A be an empty preorder. Observe that QuotientOrder(A) is empty.

Now we state the proposition:

(34) Let us consider a non empty preorder A, and an element x of A. Then $[x]_{\mathrm{EqRelOf}(A)} \in$ the carrier of QuotientOrder(A).

Let A be a non empty preorder. One can verify that QuotientOrder(A) is non empty.

Now we state the proposition:

(35) Let us consider a preorder A. Then the internal relation of QuotientOrder$(A) = \leqslant_E A$. The theorem is a consequence of (32).

Let A be a preorder. Observe that QuotientOrder(A) is reflexive, total, antisymmetric, and transitive.

Let A be a linear preorder. Let us note that QuotientOrder(A) is connected and strongly connected.

Let A be a preorder. The functor the projection onto A yielding a function from A into QuotientOrder(A) is defined by

(Def. 8) for every element x of A, $it(x) = [x]_{\mathrm{EqRelOf}(A)}$.

Let A be an empty preorder. One can check that the projection onto A is empty.

Let A be a non empty preorder. Note that the projection onto A is non empty.

Now we state the propositions:

(36) Let us consider a non empty preorder A, and elements x, y of A. Suppose $x \leqslant y$. Then (the projection onto A)(x) $\leqslant$ (the projection onto A)(y).

(37) Let us consider a preorder A, and elements x, y of A. Suppose $x \approx y$. Then (the projection onto A)(x) $=$ (the projection onto A)(y). The theorem is a consequence of (20).

Let A be a preorder and R be an equivalence relation of the carrier of A. We say that R is EqRelOf-like if and only if

(Def. 9) $R = \mathrm{EqRelOf}(A)$.

Let us note that EqRelOf(A) is EqRelOf-like and there exists an equivalence relation of the carrier of A which is EqRelOf-like.

Let R be an EqRelOf-like equivalence relation of the carrier of A and x be an element of A. One can check that the functor $[x]_R$ yields an element of QuotientOrder(A). Now we state the propositions:

(38) Let us consider a preorder A. Then the carrier of QuotientOrder(A) is a partition of the carrier of A.

(39) Let us consider a non empty preorder A, and a non empty partition D of the carrier of A. Suppose $D =$ the carrier of QuotientOrder(A). Then the projection onto $A =$ the projection onto D.

Proof: For every object x such that $x \in$ dom(the projection onto A) holds (the projection onto A)(x) $=$ (the projection onto D)(x) by [17, (23)]. $\square$

Let A be a set and D be a partition of A.

The functor PreorderFromPartition(D) yielding a strict relational structure is defined by the term

(Def. 10) $\langle A, \mathrm{EqRelOf}(D)\rangle$.

Let A be a non empty set. Let us observe that PreorderFromPartition(D) is non empty.

Let A be a set. One can verify that PreorderFromPartition(D) is reflexive and transitive and PreorderFromPartition(D) is symmetric.

Let us consider a set A and a partition D of A. Now we state the propositions:

(40) EqRelOf(D) = EqRelOf(PreorderFromPartition(D)).
PROOF: For every elements x, y of A such that $\langle x, y\rangle \in$ EqRelOf(D) holds $\langle x, y\rangle \in$ EqRelOf(PreorderFromPartition(D)) by [17, (6)]. For every elements x, y of A such that $\langle x, y\rangle \in$ EqRelOf(PreorderFromPartition(D)) holds $\langle x, y\rangle \in$ EqRelOf(D). □

(41) D = Classes(EqRelOf(PreorderFromPartition(D))). The theorem is a consequence of (40).

(42) D = the carrier of QuotientOrder(PreorderFromPartition(D)). The theorem is a consequence of (41).

Let A be a set, D be a partition of A, X be an element of D, and f be a function. The functor eqSupport(f, X) yielding a subset of A is defined by the term

(Def. 11) support $f \cap X$.

Let A be a preorder and X be an element of QuotientOrder(A). The functor eqSupport(f, X) yielding a subset of A is defined by

(Def. 12) there exists a partition D of the carrier of A and there exists an element Y of D such that D = the carrier of QuotientOrder(A) and $Y = X$ and it = eqSupport(f, Y).

Observe that the functor eqSupport(f, X) is defined by the term

(Def. 13) support $f \cap X$.

Let A be a set, D be a partition of A, f be a finite-support function, and X be an element of D. One can verify that eqSupport(f, X) is finite.

Let A be a preorder and X be an element of QuotientOrder(A). Let us note that eqSupport(f, X) is finite.

Let A be an order, X be an element of the carrier of QuotientOrder(A), and f be a finite-support function from A into $\mathbb{R}$. Observe that eqSupport(f, X) is trivial.

Now we state the propositions:

(43) Let us consider a set A, a partition D of A, an element X of D, and a function f from A into $\mathbb{R}$. Then eqSupport(f, X) = eqSupport$(-f, X)$. The theorem is a consequence of (9).

(44) Let us consider a preorder A, an element X of QuotientOrder(A), and a function f from A into $\mathbb{R}$. Then eqSupport(f, X) = eqSupport$(-f, X)$. The theorem is a consequence of (43).

Let A be a set, D be a partition of A, and f be a finite-support function from A into $\mathbb{R}$. The functor $\Sigma_D f$ yielding a function from D into $\mathbb{R}$ is defined by

(Def. 14) for every element X of D such that $X \in D$ holds $it(X) = \sum(f \cdot \mathrm{CFS}(\mathrm{eqSupport}(f, X)))$.

Let A be a preorder.

The functor $\Sigma_{\approx} f$ yielding a function from QuotientOrder(A) into $\mathbb{R}$ is defined by

(Def. 15) there exists a partition D of the carrier of A such that $D =$ the carrier of QuotientOrder(A) and $it = \Sigma_D f$.

One can verify that the functor $\Sigma_{\approx} f$ is defined by

(Def. 16) for every element X of QuotientOrder(A) such that $X \in$ the carrier of QuotientOrder(A) holds $it(X) = \sum(f \cdot \mathrm{CFS}(\mathrm{eqSupport}(f, X)))$.

Now we state the propositions:

(45) Let us consider a set A, a partition D of A, and a finite-support function f from A into $\mathbb{R}$. Then $\Sigma_D(-f) = -\Sigma_D f$.
PROOF: For every object X such that $X \in \mathrm{dom}(\Sigma_D(-f))$ holds $(\Sigma_D(-f))(X) = (-\Sigma_D f)(X)$ by (43), [1, (83)], [7, (2)], [6, (11)]. □

(46) Let us consider a preorder A, and a finite-support function f from A into $\mathbb{R}$. Then $\Sigma_{\approx} {-f} = -\Sigma_{\approx} f$. The theorem is a consequence of (38) and (45).

Let A be a preorder and f be a non-negative yielding, finite-support function from A into $\mathbb{R}$. Observe that $\Sigma_{\approx} f$ is non-negative yielding.

Let A be a set and D be a partition of A. Let us note that $\Sigma_D f$ is non-negative yielding.

Now we state the propositions:

(47) Let us consider a set A, a partition D of A, and a finite-support function f from A into $\mathbb{R}$. If f is non-positive yielding, then $\Sigma_D f$ is non-positive yielding. The theorem is a consequence of (45).

(48) Let us consider a preorder A, and a finite-support function f from A into $\mathbb{R}$. Suppose f is non-positive yielding. Then $\Sigma_{\approx} f$ is non-positive yielding. The theorem is a consequence of (38) and (47).

(49) Let us consider a preorder A, a finite-support function f from A into $\mathbb{R}$, and an element x of A. Suppose for every element y of A such that $x \approx y$ holds $x = y$. Then $(\Sigma_{\approx} f \cdot (\text{the projection onto } A))(x) = f(x)$.

(50) Let us consider an order A, and a finite-support function f from A into $\mathbb{R}$. Then $\Sigma_{\approx} f \cdot (\text{the projection onto } A) = f$.
PROOF: Set $F = \Sigma_{\approx} f \cdot (\text{the projection onto } A)$. For every object x such that $x \in \mathrm{dom}\, f$ holds $f(x) = F(x)$ by [22, (2)], (49). □

(51) Let us consider an order A, and finite-support functions f_1, f_2 from A into $\mathbb{R}$. If $\Sigma_{\approx} f_1 = \Sigma_{\approx} f_2$, then $f_1 = f_2$. The theorem is a consequence of

(50).

(52) Let us consider a preorder A, and a finite-support function f from A into $\mathbb{R}$. Then $\operatorname{support}(\Sigma_{\approx} f) \subseteq$ (the projection onto $A)^{\circ}(\operatorname{support} f)$.
PROOF: For every object X such that $X \in \operatorname{support}(\Sigma_{\approx} f)$ holds $X \in$ (the projection onto $A)^{\circ}(\operatorname{support} f)$ by [5, (24), (32)], (5), [6, (11), (13), (3)]. $\square$

(53) Let us consider a non empty set A, a non empty partition D of A, and a finite-support function f from A into $\mathbb{R}$. Then $\operatorname{support}(\Sigma_D f) \subseteq$ (the projection onto $D)^{\circ}(\operatorname{support} f)$. The theorem is a consequence of (42), (39), and (52).

(54) Let us consider a preorder A, and a finite-support function f from A into $\mathbb{R}$. Suppose f is non-negative yielding. Then (the projection onto $A)^{\circ}(\operatorname{support} f) = \operatorname{support}(\Sigma_{\approx} f)$.
PROOF:
For every object X such that $X \in$ (the projection onto $A)^{\circ}(\operatorname{support} f)$ holds $X \in \operatorname{support}(\Sigma_{\approx} f)$ by [7, (36)], [5, (24), (32)], [17, (20)]. $\square$

(55) Let us consider a non empty set A, a non empty partition D of A, and a finite-support function f from A into $\mathbb{R}$. Suppose f is non-negative yielding. Then (the projection onto $D)^{\circ}(\operatorname{support} f) = \operatorname{support}(\Sigma_D f)$. The theorem is a consequence of (42), (39), and (54).

(56) Let us consider a preorder A, and a finite-support function f from A into $\mathbb{R}$. Suppose f is non-positive yielding. Then (the projection onto $A)^{\circ}(\operatorname{support} f) = \operatorname{support}(\Sigma_{\approx} f)$. The theorem is a consequence of (9), (54), and (46).

(57) Let us consider a non empty set A, a non empty partition D of A, and a finite-support function f from A into $\mathbb{R}$. Suppose f is non-positive yielding. Then (the projection onto $D)^{\circ}(\operatorname{support} f) = \operatorname{support}(\Sigma_D f)$. The theorem is a consequence of (42), (39), and (56).

Let A be a preorder and f be a finite-support function from A into $\mathbb{R}$. Observe that $\Sigma_{\approx} f$ is finite-support.

Let A be a set and D be a partition of A. Let us note that $\Sigma_D f$ is finite-support.

Let us consider a non empty set A, a non empty partition D of A, a finite-support function f from A into $\mathbb{R}$, a one-to-one finite sequence s_1 of elements of A, and a one-to-one finite sequence s_2 of elements of D. Now we state the propositions:

(58) Suppose $\operatorname{rng} s_2 =$ (the projection onto $D)^{\circ}(\operatorname{rng} s_1)$ and for every element X of D such that $X \in \operatorname{rng} s_2$ holds $\operatorname{eqSupport}(f, X) \subseteq \operatorname{rng} s_1$. Then $\sum(\Sigma_D f \cdot s_2) = \sum(f \cdot s_1)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every one-to-one finite sequence t_1 of elements of A for every one-to-one finite sequence t_2 of elements of D such that $\operatorname{rng} t_2 = (\text{the projection onto } D)^\circ(\operatorname{rng} t_1)$ and for every element X of D such that $X \in \operatorname{rng} t_2$ holds $\operatorname{eqSupport}(f, X) \subseteq \operatorname{rng} t_1$ holds if $\operatorname{len} t_2 = \$_1$, then $\sum(\Sigma_D f \cdot t_2) = \sum(f \cdot t_1)$. $\mathcal{P}[0]$. For every natural number j such that $\mathcal{P}[j]$ holds $\mathcal{P}[j+1]$ by [5, (19)], [3, (38)], [20, (91)], [9, (48)]. For every natural number i, $\mathcal{P}[i]$ from [2, Sch. 2]. $\square$

(59) If $\operatorname{rng} s_1 = \operatorname{support} f$ and $\operatorname{rng} s_2 = \operatorname{support}(\Sigma_D f)$, then $\sum(\Sigma_D f \cdot s_2) = \sum(f \cdot s_1)$. The theorem is a consequence of (58), (53), and (8).

Now we state the proposition:

(60) Let us consider a preorder A, a finite-support function f from A into $\mathbb{R}$, a one-to-one finite sequence s_1 of elements of A, and a one-to-one finite sequence s_2 of elements of QuotientOrder(A). Suppose $\operatorname{rng} s_1 = \operatorname{support} f$ and $\operatorname{rng} s_2 = \operatorname{support}(\Sigma_{\approx} f)$. Then $\sum(\Sigma_{\approx} f \cdot s_2) = \sum(f \cdot s_1)$. The theorem is a consequence of (59).

4. Ordering Finite Sequences

Let A be a relational structure and s be a finite sequence of elements of A. We say that s is weakly ascending if and only if

(Def. 17) for every natural numbers n, m such that $n, m \in \operatorname{dom} s$ and $n < m$ holds $s_n \leqslant s_m$.

We say that s is ascending if and only if

(Def. 18) for every natural numbers n, m such that $n, m \in \operatorname{dom} s$ and $n < m$ holds $s_n \lnapprox s_m$.

Let us observe that every finite sequence of elements of A which is ascending is also weakly ascending.

Let A be an antisymmetric relational structure and s be a finite sequence of elements of A. Observe that s is ascending if and only if the condition (Def. 19) is satisfied.

(Def. 19) for every natural numbers n, m such that $n, m \in \operatorname{dom} s$ and $n < m$ holds $s_n < s_m$.

Let A be a relational structure. We say that s is weakly descending if and only if

(Def. 20) for every natural numbers n, m such that $n, m \in \operatorname{dom} s$ and $n < m$ holds $s_m \leqslant s_n$.

We say that s is descending if and only if

(Def. 21) for every natural numbers n, m such that $n, m \in \operatorname{dom} s$ and $n < m$ holds $s_m \lnsim s_n$.

One can verify that every finite sequence of elements of A which is descending is also weakly descending.

Let A be an antisymmetric relational structure and s be a finite sequence of elements of A. Let us observe that s is descending if and only if the condition (Def. 22) is satisfied.

(Def. 22) for every natural numbers n, m such that $n, m \in \operatorname{dom} s$ and $n < m$ holds $s_m < s_n$.

Note that every finite sequence of elements of A which is one-to-one and weakly ascending is also ascending and every finite sequence of elements of A which is one-to-one and weakly descending is also descending and every finite sequence of elements of A which is weakly ascending and weakly descending is also constant.

Let A be a reflexive relational structure. Note that every finite sequence of elements of A which is constant is also weakly ascending and weakly descending.

Let A be a relational structure. Note that $\varepsilon_{(\text{the carrier of } A)}$ is ascending, weakly ascending, descending, and weakly descending and there exists a finite sequence of elements of A which is empty, ascending, weakly ascending, descending, and weakly descending.

Let A be a non empty relational structure and x be an element of A. Let us observe that $\langle x \rangle$ is ascending, weakly ascending, descending, and weakly descending as a finite sequence of elements of A and there exists a finite sequence of elements of A which is non empty, one-to-one, ascending, weakly ascending, descending, and weakly descending.

Let A be a relational structure and s be a finite sequence of elements of A. We say that s is asc-ordering if and only if

(Def. 23) s is one-to-one and weakly ascending.

We say that s is desc-ordering if and only if

(Def. 24) s is one-to-one and weakly descending.

Let us note that every finite sequence of elements of A which is asc-ordering is also one-to-one and weakly ascending and every finite sequence of elements of A which is one-to-one and weakly ascending is also asc-ordering and every finite sequence of elements of A which is desc-ordering is also one-to-one and weakly descending and every finite sequence of elements of A which is one-to-one and weakly descending is also desc-ordering and every finite sequence of elements of A which is ascending is also asc-ordering and every finite sequence of elements of A which is descending is also desc-ordering.

Let B be a subset of A and s be a finite sequence of elements of A. We say that s is B-asc-ordering if and only if

(Def. 25) s is asc-ordering and $\operatorname{rng} s = B$.

We say that s is B-desc-ordering if and only if

(Def. 26) s is desc-ordering and $\operatorname{rng} s = B$.

Let us observe that every finite sequence of elements of A which is B-asc-ordering is also asc-ordering and every finite sequence of elements of A which is B-desc-ordering is also desc-ordering.

Let B be an empty subset of A. Let us note that every finite sequence of elements of A which is B-asc-ordering is also empty and every finite sequence of elements of A which is B-desc-ordering is also empty.

Let us consider a relational structure A and a finite sequence s of elements of A. Now we state the propositions:

(61) s is weakly ascending if and only if $\operatorname{Rev}(s)$ is weakly descending.

(62) s is ascending if and only if $\operatorname{Rev}(s)$ is descending.

Let us consider a relational structure A, a subset B of A, and a finite sequence s of elements of A. Now we state the propositions:

(63) s is B-asc-ordering if and only if $\operatorname{Rev}(s)$ is B-desc-ordering. The theorem is a consequence of (61).

(64) If s is B-asc-ordering or B-desc-ordering, then B is finite.

Let A be an antisymmetric relational structure. One can check that every finite sequence of elements of A which is asc-ordering is also ascending and every finite sequence of elements of A which is desc-ordering is also descending.

Let us consider an antisymmetric relational structure A, a subset B of A, and finite sequences s_1, s_2 of elements of A. Now we state the propositions:

(65) If s_1 is B-asc-ordering and s_2 is B-asc-ordering, then $s_1 = s_2$.
Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \in \operatorname{dom} s_1$ and $\$_1 \in \operatorname{dom} s_2$, then $s_{1\$_1} = s_{2\$_1}$. For every natural number k such that for every natural number n such that $n < k$ holds $\mathcal{P}[n]$ holds $\mathcal{P}[k]$ by [5, (10)], [22, (2)]. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 4]. For every natural number k such that $k \in \operatorname{dom} s_1$ holds $s_1(k) = s_2(k)$. $\square$

(66) If s_1 is B-desc-ordering and s_2 is B-desc-ordering, then $s_1 = s_2$. The theorem is a consequence of (63) and (65).

(67) Let us consider a linear order A, a finite subset B of A, and a finite sequence s of elements of A. Then s is B-asc-ordering if and only if $s = \operatorname{SgmX}((\text{the internal relation of } A), B)$.
Proof: If s is B-asc-ordering, then $s = \operatorname{SgmX}((\text{the internal relation of } A), B)$ by [8, (4)]. The internal relation of A linearly orders B. For every

natural numbers n, m such that n, $m \in \operatorname{dom} s$ and $n < m$ holds $s_n < s_m$. $\square$

Let A be a linear order and B be a finite subset of A.

Observe that SgmX((the internal relation of A), B) is B-asc-ordering.

Let us consider a relational structure A, subsets B, C of A, and a finite sequence s of elements of A. Now we state the propositions:

(68) If s is B-asc-ordering and $C \subseteq B$, then there exists a finite sequence s_2 of elements of A such that s_2 is C-asc-ordering.

PROOF: Set $s_2 = s \cdot \operatorname{Sgm}(s^{-1}(C))$. Consider n being a natural number such that $\operatorname{dom} s = \operatorname{Seg} n$. For every object x, $x \in \operatorname{rng} s_2$ iff $x \in C$ by [6, (11), (3), (12)]. For every natural numbers n, m such that n, $m \in \operatorname{dom} s_2$ and $n < m$ holds $s_{2n} \leqslant s_{2m}$ by [3, (6)], [6, (11)], [3, (1)], [6, (12)]. $\square$

(69) If s is B-desc-ordering and $C \subseteq B$, then there exists a finite sequence s_2 of elements of A such that s_2 is C-desc-ordering. The theorem is a consequence of (63) and (68).

(70) Let us consider a relational structure A, a subset B of A, a finite sequence s of elements of A, and an element x of A. Suppose $B = \{x\}$ and $s = \langle x \rangle$. Then s is B-asc-ordering and B-desc-ordering.

PROOF: For every natural numbers n, m such that n, $m \in \operatorname{dom} s$ and $n < m$ holds $s_n \leqslant s_m \leqslant s_n$ by [3, (38), (2)]. $\square$

Let us consider a relational structure A, a subset B of A, and a finite sequence s of elements of A. Now we state the propositions:

(71) If s is B-asc-ordering, then the internal relation of A is connected in B.

PROOF: For every objects x, y such that x, $y \in B$ and $x \neq y$ holds $\langle x, y \rangle \in$ the internal relation of A or $\langle y, x \rangle \in$ the internal relation of A by [5, (10)]. $\square$

(72) If s is B-desc-ordering, then the internal relation of A is connected in B. The theorem is a consequence of (63) and (71).

Let us consider a transitive relational structure A, subsets B, C of A, a finite sequence s_1 of elements of A, and an element x of A. Now we state the propositions:

(73) Suppose s_1 is C-asc-ordering and $x \notin C$ and $B = C \cup \{x\}$ and for every element y of A such that $y \in C$ holds $x \leqslant y$. Then there exists a finite sequence s_2 of elements of A such that

(i) $s_2 = \langle x \rangle \frown s_1$, and

(ii) s_2 is B-asc-ordering.

PROOF: Set $s_3 = \langle x \rangle$. Set $s_2 = s_3 \frown s_1$. For every natural numbers n, m such that n, $m \in \operatorname{dom} s_2$ and $n < m$ holds $s_{2n} \leqslant s_{2m}$ by [3, (25), (38), (2)]. □

(74) Suppose s_1 is C-asc-ordering and $x \notin C$ and $B = C \cup \{x\}$ and for every element y of A such that $y \in C$ holds $y \leqslant x$. Then there exists a finite sequence s_2 of elements of A such that

(i) $s_2 = s_1 \frown \langle x \rangle$, and

(ii) s_2 is B-asc-ordering.

Proof: Set $s_3 = \langle x \rangle$. Set $s_2 = s_1 \frown s_3$. For every natural numbers n, m such that $n, m \in \operatorname{dom} s_2$ and $n < m$ holds $s_{2n} \leqslant s_{2m}$ by [3, (25), (1), (2)], [2, (13)]. □

(75) Suppose s_1 is C-desc-ordering and $x \notin C$ and $B = C \cup \{x\}$ and for every element y of A such that $y \in C$ holds $x \leqslant y$. Then there exists a finite sequence s_2 of elements of A such that

(i) $s_2 = s_1 \frown \langle x \rangle$, and

(ii) s_2 is B-desc-ordering.

The theorem is a consequence of (63) and (73).

(76) Suppose s_1 is C-desc-ordering and $x \notin C$ and $B = C \cup \{x\}$ and for every element y of A such that $y \in C$ holds $y \leqslant x$. Then there exists a finite sequence s_2 of elements of A such that

(i) $s_2 = \langle x \rangle \frown s_1$, and

(ii) s_2 is B-desc-ordering.

The theorem is a consequence of (63) and (74).

Let us consider a transitive relational structure A and a finite subset B of A. Now we state the propositions:

(77) If the internal relation of A is connected in B, then there exists a finite sequence s of elements of A such that s is B-asc-ordering.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every subset C of A such that $C \subseteq B$ and $\overline{\overline{C}} = \$_1$ there exists a finite sequence s of elements of A such that s is C-asc-ordering. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by (2), [3, (74)], (70), (22). For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. □

(78) If the internal relation of A is connected in B, then there exists a finite sequence s of elements of A such that s is B-desc-ordering. The theorem is a consequence of (77) and (63).

Let us consider a connected, transitive relational structure A and a finite subset B of A. Now we state the propositions:

(79) There exists a finite sequence s of elements of A such that s is B-asc-ordering. The theorem is a consequence of (77).

(80) There exists a finite sequence s of elements of A such that s is B-desc-ordering. The theorem is a consequence of (79) and (63).

Let A be a connected, transitive relational structure and B be a finite subset of A. Note that there exists a finite sequence of elements of A which is B-asc-ordering and there exists a finite sequence of elements of A which is B-desc-ordering.

Now we state the proposition:

(81) Let us consider a preorder A, and a subset B of A. Suppose the internal relation of A is connected in B. Then the internal relation of QuotientOrder(A) is connected in (the projection onto $A)^\circ B$. The theorem is a consequence of (36).

Let us consider a preorder A, a subset B of A, and a finite sequence s_1 of elements of A. Now we state the propositions:

(82) Suppose s_1 is B-asc-ordering. Then there exists a finite sequence s_2 of elements of QuotientOrder(A) such that s_2 is ((the projection onto $A)^\circ B$)-asc-ordering. The theorem is a consequence of (71), (81), and (77).

(83) Suppose s_1 is B-desc-ordering. Then there exists a finite sequence s_2 of elements of QuotientOrder(A) such that s_2 is ((the projection onto $A)^\circ B$)-desc-ordering. The theorem is a consequence of (63) and (82).

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