

Basel Problem – Preliminaries

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Summary. In the article we formalize in the Mizar system [4] preliminary facts needed to prove the Basel problem [7, 1]. Facts that are independent from the notion of structure are included here.

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1. PRELIMINARIES

From now on X denotes a set, k, m, n denote natural numbers, i denotes an integer, $a, b, c, d, e, g, p, r, x, y$ denote real numbers, and z denotes a complex.

Now we state the proposition:

- (1) If $0 < a$, then there exists m such that $0 < a \cdot m + b$.

Let f be a real-valued finite sequence. Let us consider n . Observe that $f \upharpoonright n$ is \mathbb{R} -valued.

Let f be a complex-valued finite sequence. Let us observe that f^2 is $(\text{len } f)$ -element and f^{-1} is $(\text{len } f)$ -element.

Let c be a complex. Note that $c + f$ is $(\text{len } f)$ -element.

Now we state the propositions:

- (2) Let us consider complexes c, z . Then $c + \langle z \rangle = \langle c + z \rangle$.

- (3) Let us consider complex-valued finite sequences f, g , and a complex c .

Then $c + f \wedge g = (c + f) \wedge (c + g)$.

- (4) Let us consider a complex-valued finite sequence f , and a complex c . Then $\sum(c + f) = c \cdot \text{len } f + \sum f$.

PROOF: Define $\mathcal{P}[\text{complex-valued finite sequence}] \equiv \sum(c + \$1) = c \cdot \text{len } \$1 + \sum \$1$. For every finite sequence p of elements of \mathbb{C} and for every element x of \mathbb{C} such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \hat{\ } \langle x \rangle]$ by [3, (39), (22)], (2), [17, (32)]. For every finite sequence p of elements of \mathbb{C} , $\mathcal{P}[p]$ from [5, Sch. 2]. \square

2. LIMITS OF SEQUENCES $\frac{an+b}{cn+d}$

Let a, b, c, d be complexes. The functor $\text{Rat-Exp-Seq}(a, b, c, d)$ yielding a complex sequence is defined by

(Def. 1) $it(n) = \frac{\text{Polynom}(a, b, n)}{\text{Polynom}(c, d, n)}$.

Let us consider a, b, c , and d . The functor $\text{rseq}(a, b, c, d)$ yielding a sequence of real numbers is defined by the term

(Def. 2) $\Re(\text{Rat-Exp-Seq}(a, b, c, d))$.

Now we state the propositions:

(5) $(\text{rseq}(a, b, c, d))(n) = \frac{a \cdot n + b}{c \cdot n + d}$.

(6) $(\text{rseq}(0, b, 0, d))(n) = \frac{b}{d}$. The theorem is a consequence of (5).

Let us consider a and b . Let us note that $\text{rseq}(a, b, 0, 0)$ is constant.

Let us consider d . One can verify that $\text{rseq}(0, b, 0, d)$ is constant.

Now we state the propositions:

(7) (i) $\text{rseq}(0, b, c, d) = b \cdot \text{rseq}(0, 1, c, d)$, and

(ii) $\text{rseq}(0, b, c, d) = (-b) \cdot \text{rseq}(0, 1, -c, -d)$.

The theorem is a consequence of (5).

(8) (i) $\text{rseq}(a, 0, c, d) = a \cdot \text{rseq}(1, 0, c, d)$, and

(ii) $\text{rseq}(a, 0, c, d) = (-a) \cdot \text{rseq}(1, 0, -c, -d)$.

The theorem is a consequence of (5).

Let us consider b, c , and d . Let us observe that $\text{rseq}(0, b, c, d)$ is convergent.

Now we state the propositions:

(9) $\lim \text{rseq}(0, b, 0, d) = \frac{b}{d}$. The theorem is a consequence of (6).

(10) Let us consider a non zero real number c . Then $\lim \text{rseq}(0, b, c, d) = 0$.

The theorem is a consequence of (5).

Let c be a non zero real number. Let us consider a, b , and d . Note that $\text{rseq}(a, b, c, d)$ is convergent.

Let a, d be positive real numbers and b be a real number. Let us observe that $\text{rseq}(a, b, 0, d)$ is non upper bounded.

Let a, d be negative real numbers. Let us consider b . One can check that $\text{rseq}(a, b, 0, d)$ is non upper bounded.

Let a be a positive real number and d be a negative real number. Note that $\text{rseq}(a, b, 0, d)$ is non lower bounded.

Let a be a negative real number and d be a positive real number. Let us note that $\text{rseq}(a, b, 0, d)$ is non lower bounded.

Let a, d be non zero real numbers. One can check that $\text{rseq}(a, b, 0, d)$ is non bounded and $\text{rseq}(a, b, 0, d)$ is non convergent.

Now we state the propositions:

- (11) Let us consider a non zero real number c . Then $\lim \text{rseq}(a, b, c, d) = \frac{a}{c}$.
The theorem is a consequence of (5) and (10).
- (12) Let us consider a non zero real number a . Then $\lim \text{rseq}(a, b, a, d) = 1$.
The theorem is a consequence of (11).

3. TRIGONOMETRY

Now we state the propositions:

- (13) $\sin(\pi \cdot i) = 0$.
- (14) $\cos(\frac{\pi}{2} + (\pi \cdot i)) = 0$.
- (15) (i) $\tan r = (\cot r)^{-1}$, and
(ii) $\cot r = (\tan r)^{-1}$.
- (16) $\text{dom}(\text{the function } \tan) = \bigcup \text{the set of all }]-\frac{\pi}{2} + (\pi \cdot i), \frac{\pi}{2} + (\pi \cdot i)[$ where i is an integer.

PROOF: Set $S = \text{the set of all }]-\frac{\pi}{2} + (\pi \cdot i), \frac{\pi}{2} + (\pi \cdot i)[$ where i is an integer. Set $T = \text{dom}(\text{the function } \tan)$. $T \subseteq \bigcup S$ by (14), [24, (29)]. For every set X such that $X \in S$ holds $X \subseteq T$ by [16, (11)], [8, (9)], [21, (1)], [16, (13)].
□

Observe that $\text{dom}(\text{the function } \tan)$ is open as a subset of \mathbb{R} .

Now we state the propositions:

- (17) If $0 \leq r$, then $(\text{the function } \sin)(r) \leq r$.
PROOF: Reconsider $A = [0, r]$ as a non empty, closed interval subset of \mathbb{R} . Reconsider $c = (\text{the function } \cos) \upharpoonright A$ as a function from A into \mathbb{R} . $c \upharpoonright A$ is bounded and c is integrable by [11, (11), (10)]. $\text{integral } c = (\text{the function } \sin)(r)$ by [11, (19)], [22, (24)], [26, (30)]. Set $Z_0 = \square^0$. Reconsider $Z_3 = Z_0 \upharpoonright A$ as a function from A into \mathbb{R} . $Z_3 \upharpoonright A$ is bounded and Z_3 is integrable by [11, (11), (10)]. $\text{integral } Z_3 = r$ by [14, (21)], [19, (35)], [11, (19)], [22, (30)]. For every r such that $r \in A$ holds $c(r) \leq Z_3(r)$ by [6, (49)], [19, (34)], [13, (6)]. □

(18) If $0 \leq r < \frac{\pi}{2}$, then $r \leq (\text{the function tan})(r)$.

PROOF: Reconsider $A = [0, r]$ as a non empty, closed interval subset of \mathbb{R} . Set $Z_0 = \square^0$. Reconsider $Z_3 = Z_0 \upharpoonright A$ as a function from A into \mathbb{R} . $Z_3 \upharpoonright A$ is bounded and Z_3 is integrable by [11, (11), (10)]. integral $Z_3 = r$ by [14, (21)], [19, (35)], [11, (19)], [22, (30)]. Set $T = \text{dom}(\text{the function tan})$. Set $c_2 = (\text{the function cos}) \cdot (\text{the function cos})$. Set $c_3 = c_2 \upharpoonright T$. Set $Z_1 = \frac{Z_0}{c_3}$. $c_3^{-1}(\{0\}) = \emptyset$ by [6, (47)]. Reconsider $Z_2 = Z_1 \upharpoonright A$ as a function from A into \mathbb{R} . $Z_1 \upharpoonright A$ is continuous and $Z_2 \upharpoonright A$ is bounded and Z_2 is integrable by [20, (24)], [11, (11), (10)]. For every real number s such that $s \in T$ holds $Z_1(s) = \frac{1}{(\text{the function cos})(s)^2}$ and $(\text{the function cos})(s) \neq 0$ by [19, (34)], [6, (47)]. integral $Z_2 = (\text{the function tan})(r)$ by [12, (19)], [18, (59)], [15, (41)]. For every r such that $r \in A$ holds $Z_3(r) \leq Z_2(r)$ by [6, (49)], [19, (34)], [16, (11)], [13, (6)]. \square

4. SOME SPECIAL FUNCTIONS AND SEQUENCES

Let f be a real-valued function. The functors: $\cot f$ and $\text{cosec } f$ yielding functions are defined by conditions

(Def. 3) $\text{dom } \cot f = \text{dom } f$ and for every object x such that $x \in \text{dom } f$ holds $\cot f(x) = \cot(f(x))$,

(Def. 4) $\text{dom } \text{cosec } f = \text{dom } f$ and for every object x such that $x \in \text{dom } f$ holds $\text{cosec } f(x) = \text{cosec}(f(x))$,

respectively. Note that $\cot f$ is \mathbb{R} -valued and $\text{cosec } f$ is \mathbb{R} -valued.

Let f be a real-valued finite sequence. Let us observe that $\cot f$ is finite sequence-like and $\text{cosec } f$ is finite sequence-like.

Let us consider a real-valued finite sequence f . Now we state the propositions:

(19) $\text{len } \cot f = \text{len } f$.

(20) $\text{len } \text{cosec } f = \text{len } f$.

Let f be a real-valued finite sequence. Note that $\cot f$ is $(\text{len } f)$ -element and $\text{cosec } f$ is $(\text{len } f)$ -element.

Let us consider m . The functor $\text{x-r-seq}(m)$ yielding a finite sequence is defined by the term

(Def. 5) $\frac{\pi}{2 \cdot m + 1} \cdot \text{idseq}(m)$.

Now we state the propositions:

(21) (i) $\text{len } \text{x-r-seq}(m) = m$, and

(ii) for every k such that $1 \leq k \leq m$ holds $(\text{x-r-seq}(m))(k) = \frac{k \cdot \pi}{2 \cdot m + 1}$.

(22) $\text{rng } \text{x-r-seq}(m) \subseteq]0, \frac{\pi}{2}[$. The theorem is a consequence of (21).

Let us consider m . Let us note that $x\text{-r-seq}(m)$ is \mathbb{R} -valued.

Now we state the proposition:

(23) If $1 \leq k \leq m$, then $0 < (x\text{-r-seq}(m))(k) < \frac{\pi}{2}$. The theorem is a consequence of (22) and (21).

Note that $x\text{-r-seq}(0)$ is empty.

(24) If $1 \leq k \leq m$, then $\frac{2}{k \cdot \pi} + (x\text{-r-seq}(m))^{-1}(k) = (x\text{-r-seq}(m+1))^{-1}(k)$. The theorem is a consequence of (21).

(25) If $1 \leq k \leq m$, then $2 \cdot m + 1 \cdot (x\text{-r-seq}(m))(k) = k \cdot \pi$. The theorem is a consequence of (21).

(26) ${}^2\text{cosec } x\text{-r-seq}(m) = 1 + {}^2\text{cot } x\text{-r-seq}(m)$. The theorem is a consequence of (21) and (23).

(27) $x\text{-r-seq}(n)$ is one-to-one. The theorem is a consequence of (21).

(28) ${}^2\text{cot } x\text{-r-seq}(n)$ is one-to-one.

PROOF: Set $f = x\text{-r-seq}(n)$. f is one-to-one. $0 < f(x_1) < \frac{\pi}{2}$ and $0 < f(x_2) < \frac{\pi}{2}$ and $\frac{\pi}{2} < \pi$. $\text{cot}(f(x_1)) = \text{cot}(f(x_2))$ by [23, (40)]. $f(x_1) = f(x_2)$ by [15, (2)], [25, (57)], [6, (47)], [15, (10)]. \square

(29) $\sum({}^2\text{cot } x\text{-r-seq}(m)) \leq \sum({}^2x\text{-r-seq}(m))^{-1}$. The theorem is a consequence of (21), (19), (15), (23), (16), and (18).

(30) $\sum({}^2x\text{-r-seq}(m))^{-1} \leq \sum({}^2\text{cosec } x\text{-r-seq}(m))$. The theorem is a consequence of (21), (20), (23), and (17).

The functors: Basel-seq , Basel-seq^1 , and Basel-seq^2 yielding sequences of real numbers are defined by terms

(Def. 6) $\text{rseq}(0, 1, 1, 0) \cdot \text{rseq}(0, 1, 1, 0)$,

(Def. 7) $(\frac{\pi^2}{6} \cdot \text{rseq}(2, 0, 2, 1)) \cdot \text{rseq}(2, -1, 2, 1)$,

(Def. 8) $(\frac{\pi^2}{6} \cdot \text{rseq}(2, 0, 2, 1)) \cdot \text{rseq}(2, 2, 2, 1)$,

respectively. Now we state the propositions:

(31) $(\text{Basel-seq})(n) = \frac{1}{n^2}$.

(32) $(\text{Basel-seq}^1)(n) = \frac{\pi^2}{6} \cdot \frac{2 \cdot n}{2 \cdot n + 1} \cdot \frac{2 \cdot n - 1}{2 \cdot n + 1}$. The theorem is a consequence of (5).

(33) $(\text{Basel-seq}^2)(n) = \frac{\pi^2}{6} \cdot \frac{2 \cdot n}{2 \cdot n + 1} \cdot \frac{2 \cdot n + 2}{2 \cdot n + 1}$. The theorem is a consequence of (5).

Let us observe that Basel-seq is convergent and Basel-seq^1 is convergent and Basel-seq^2 is convergent.

(34) $\lim \text{Basel-seq}^1 = \frac{\pi^2}{6} = \lim \text{Basel-seq}^2$.

(35) $\sum({}^2x\text{-r-seq}(m))^{-1} = \frac{(2 \cdot m + 1)^2}{\pi^2} \cdot \sum_{\kappa=0}^m \text{Basel-seq}(\kappa)$.

PROOF: Set $a = \pi^2$. Set $b = (2 \cdot m + 1)^2$. Set $B = \text{Basel-seq}$. Set $S = \text{Shift}(B \upharpoonright \mathbb{Z}_{m+1}, 1)$. Set $M = x\text{-r-seq}(m)$. Set $G = ({}^2M)^{-1}$. Set $F = \langle 0 \rangle \wedge G$. $B(0) = \frac{1}{0^2}$. $F = \frac{b}{a} \cdot S$ by [9, (3)], [2, (11)], [10, (47)], (31). \square

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