

# Niven's Theorem<sup>1</sup>

Artur Kornilowicz  
Institute of Informatics  
University of Białystok  
Poland

Adam Naumowicz  
Institute of Informatics  
University of Białystok  
Poland

**Summary.** This article formalizes the proof of Niven's theorem [12] which states that if  $x/\pi$  and  $\sin(x)$  are both rational, then the sine takes values 0,  $\pm 1/2$ , and  $\pm 1$ . The main part of the formalization follows the informal proof presented at ProofWiki ([https://proofwiki.org/wiki/Niven's\\_Theorem#Source\\_of\\_Name](https://proofwiki.org/wiki/Niven's_Theorem#Source_of_Name)). For this proof, we have also formalized the rational and integral root theorems setting constraints on solutions of polynomial equations with integer coefficients [8, 9].

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From now on  $r, t$  denote real numbers,  $i$  denotes an integer,  $k, n$  denote natural numbers,  $p$  denotes a polynomial over  $\mathbb{R}_F$ ,  $e$  denotes an element of  $\mathbb{R}_F$ ,  $L$  denotes a non empty zero structure, and  $z, z_0, z_1, z_2$  denote elements of  $L$ .

Now we state the propositions:

- (1) Let us consider complexes  $a, b, c, d$ . If  $b \neq 0$  and  $\frac{a}{b} = \frac{c}{d}$ , then  $a = \frac{b \cdot c}{d}$ .
- (2) Let us consider real numbers  $a, b$ . If  $|a| = b$ , then  $a = b$  or  $a = -b$ .
- (3) If  $|i| \leq 2$ , then  $i = -2$  or  $i = -1$  or  $i = 0$  or  $i = 1$  or  $i = 2$ . The theorem is a consequence of (2).
- (4) If  $n \neq 0$ , then  $i \mid i^n$ .
- (5) If  $t > 0$ , then there exists  $i$  such that  $t \cdot i \leq r \leq t \cdot (i + 1)$ .

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PROOF: Define  $\mathcal{P}[\text{integer}] \equiv t \cdot \$1 \leq r$ . There exists an integer  $i_1$  such that  $\mathcal{P}[i_1]$ . Set  $F = \lceil \frac{r}{t} \rceil$ . For every integer  $i_1$  such that  $\mathcal{P}[i_1]$  holds  $i_1 \leq F$ . Consider  $i$  such that  $\mathcal{P}[i]$  and for every integer  $i_1$  such that  $\mathcal{P}[i_1]$  holds  $i_1 \leq i$  from [15, Sch. 6].  $\square$

- (6) Let us consider a finite sequence  $p$  of elements of  $\mathbb{R}_F$ , and a real-valued finite sequence  $q$ . If  $p = q$ , then  $\sum p = \sum q$ .

PROOF: Define  $\mathcal{P}[\text{finite sequence}] \equiv$  for every finite sequence  $p$  of elements of  $\mathbb{R}_F$  for every real-valued finite sequence  $q$  such that  $p = q$  and  $p = \$1$  holds  $\sum p = \sum q$ .  $\mathcal{P}[\emptyset]$  by [16, (43)], [4, (72)]. For every finite sequence  $f$  and for every object  $x$  such that  $\mathcal{P}[f]$  holds  $\mathcal{P}[f \cap \langle x \rangle]$  by [2, (36), (38)], [5, (31)], [16, (41), (44)]. For every finite sequence  $f$ ,  $\mathcal{P}[f]$  from [2, Sch. 3].  $\square$

- (7) Let us consider a natural number  $i$ , and an element  $r$  of  $\mathbb{R}_F$ . Then  $\text{power}_{\mathbb{R}_F}(r, i) = r^i$ .

PROOF: Define  $\mathcal{P}[\text{natural number}] \equiv \text{power}_{\mathbb{R}_F}(r, \$1) = r^{\$1}$ . For every natural number  $n$ ,  $\mathcal{P}[n]$  from [1, Sch. 2].  $\square$

- (8)  $\sin(\frac{5 \cdot \pi}{6}) = \frac{1}{2}$ .  
(9)  $\sin(\frac{5 \cdot \pi}{6} + 2 \cdot \pi \cdot i) = \frac{1}{2}$ .  
(10)  $\sin(\frac{7 \cdot \pi}{6}) = -\frac{1}{2}$ .  
(11)  $\sin(\frac{7 \cdot \pi}{6} + 2 \cdot \pi \cdot i) = -\frac{1}{2}$ .  
(12)  $\sin(\frac{11 \cdot \pi}{6}) = -\frac{1}{2}$ .  
(13)  $\sin(\frac{11 \cdot \pi}{6} + 2 \cdot \pi \cdot i) = -\frac{1}{2}$ .  
(14)  $\cos(\frac{4 \cdot \pi}{3}) = -\frac{1}{2}$ .  
(15)  $\cos(\frac{4 \cdot \pi}{3} + 2 \cdot \pi \cdot i) = -\frac{1}{2}$ .  
(16)  $\cos(\frac{5 \cdot \pi}{3}) = \frac{1}{2}$ .  
(17)  $\cos(\frac{5 \cdot \pi}{3} + 2 \cdot \pi \cdot i) = \frac{1}{2}$ .  
(18) If  $0 \leq r \leq \frac{\pi}{2}$  and  $\cos r = \frac{1}{2}$ , then  $r = \frac{\pi}{3}$ .  
(19) Let us consider an add-associative, right zeroed, right complementable, left distributive, non empty double loop structure  $L$ , and a sequence  $p$  of  $L$ . Then  $\mathbf{0} \cdot L * p = \mathbf{0} \cdot L$ .

Let us consider  $L$ ,  $z$ , and  $n$ . One can verify that  $\mathbf{0} \cdot L + \cdot (n, z)$  is finite-Support as a sequence of  $L$ .

Let us consider a polynomial  $p$  over  $L$ . Now we state the propositions:

- (20) If  $z \neq 0_L$ , then if  $p = \mathbf{0} \cdot L + \cdot (n, z)$ , then  $\text{len } p = n + 1$ .

PROOF: the length of  $p$  is at most  $n + 1$  by [1, (13)], [3, (32)], [14, (7)]. For every natural number  $m$  such that the length of  $p$  is at most  $m$  holds  $n + 1 \leq m$  by [14, (13)], [3, (31)], [1, (13)].  $\square$

(21) If  $z \neq 0_L$ , then if  $p = \mathbf{0}.L + \cdot (n, z)$ , then  $\text{deg } p = n$ . The theorem is a consequence of (20).

Note that  $\mathbf{0}. \mathbb{R}_F$  is  $\mathbb{Z}$ -valued and  $\mathbf{1}. \mathbb{R}_F$  is  $\mathbb{Z}$ -valued and there exists an element of  $\mathbb{R}_F$  which is integer.

Now we state the proposition:

(22)  $\text{rng}\langle z \rangle = \{z, 0_L\}$ .

PROOF: Set  $p = \langle z \rangle$ .  $\text{rng } p \subseteq \{z, 0_L\}$  by [11, (32)], [1, (14)].  $\square$

Let us consider  $L$ ,  $z_0$ ,  $z_1$ , and  $z_2$ . The functor  $\langle z_0, z_1, z_2 \rangle$  yielding a sequence of  $L$  is defined by the term

(Def. 1)  $((\mathbf{0}.L + \cdot (0, z_0)) + \cdot (1, z_1)) + \cdot (2, z_2)$ .

Now we state the propositions:

(23)  $\langle z_0, z_1, z_2 \rangle(0) = z_0$ .

(24)  $\langle z_0, z_1, z_2 \rangle(1) = z_1$ .

(25)  $\langle z_0, z_1, z_2 \rangle(2) = z_2$ .

(26) If  $3 \leq n$ , then  $\langle z_0, z_1, z_2 \rangle(n) = 0_L$ .

Let us consider  $L$ ,  $z_0$ ,  $z_1$ , and  $z_2$ . Let us observe that  $\langle z_0, z_1, z_2 \rangle$  is finite-Support.

Now we state the propositions:

(27)  $\text{len}\langle z_0, z_1, z_2 \rangle \leq 3$ . The theorem is a consequence of (26).

(28) If  $z_2 \neq 0_L$ , then  $\text{len}\langle z_0, z_1, z_2 \rangle = 3$ . The theorem is a consequence of (25) and (26).

(29) Let us consider a right zeroed, non empty additive loop structure  $L$ , and elements  $z_0, z_1$  of  $L$ . Then  $\langle z_0 \rangle + \langle z_1 \rangle = \langle z_0 + z_1 \rangle$ .

(30) Let us consider a right zeroed, non empty additive loop structure  $L$ , and elements  $z_0, z_1, z_2, z_3$  of  $L$ . Then  $\langle z_0, z_1 \rangle + \langle z_2, z_3 \rangle = \langle z_0 + z_2, z_1 + z_3 \rangle$ .

(31) Let us consider a right zeroed, non empty additive loop structure  $L$ , and elements  $z_0, z_1, z_2, z_3, z_4, z_5$  of  $L$ . Then  $\langle z_0, z_1, z_2 \rangle + \langle z_3, z_4, z_5 \rangle = \langle z_0 + z_3, z_1 + z_4, z_2 + z_5 \rangle$ . The theorem is a consequence of (23), (24), (25), and (26).

(32) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure  $L$ , and an element  $z_0$  of  $L$ . Then  $-\langle z_0 \rangle = \langle -z_0 \rangle$ .

(33) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure  $L$ , and elements  $z_0, z_1$  of  $L$ . Then  $-\langle z_0, z_1 \rangle = \langle -z_0, -z_1 \rangle$ .

(34) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure  $L$ , and elements  $z_0, z_1, z_2$  of  $L$ . Then

- $-\langle z_0, z_1, z_2 \rangle = \langle -z_0, -z_1, -z_2 \rangle$ . The theorem is a consequence of (23), (24), (25), and (26).
- (35) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure  $L$ , and elements  $z_0, z_1$  of  $L$ . Then  $\langle z_0 \rangle - \langle z_1 \rangle = \langle z_0 - z_1 \rangle$ . The theorem is a consequence of (32) and (29).
- (36) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure  $L$ , and elements  $z_0, z_1, z_2, z_3$  of  $L$ . Then  $\langle z_0, z_1 \rangle - \langle z_2, z_3 \rangle = \langle z_0 - z_2, z_1 - z_3 \rangle$ . The theorem is a consequence of (33) and (30).
- (37) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure  $L$ , and elements  $z_0, z_1, z_2, z_3, z_4, z_5$  of  $L$ . Then  $\langle z_0, z_1, z_2 \rangle - \langle z_3, z_4, z_5 \rangle = \langle z_0 - z_3, z_1 - z_4, z_2 - z_5 \rangle$ . The theorem is a consequence of (34) and (31).
- (38) Let us consider an add-associative, right zeroed, right complementable, left distributive, unital, associative, non empty double loop structure  $L$ , and elements  $z_0, z_1, z_2, x$  of  $L$ . Then  $\text{eval}(\langle z_0, z_1, z_2 \rangle, x) = z_0 + z_1 \cdot x + z_2 \cdot x \cdot x$ . The theorem is a consequence of (23), (24), (27), and (25).

Let  $a$  be an integer element of  $\mathbb{R}_F$ . Note that  $\langle a \rangle$  is  $\mathbb{Z}$ -valued.

Let  $a, b$  be integer elements of  $\mathbb{R}_F$ . One can verify that  $\langle a, b \rangle$  is  $\mathbb{Z}$ -valued.

Let  $a, b, c$  be integer elements of  $\mathbb{R}_F$ . Observe that  $\langle a, b, c \rangle$  is  $\mathbb{Z}$ -valued and there exists a polynomial over  $\mathbb{R}_F$  which is monic and  $\mathbb{Z}$ -valued and there exists a finite sequence of elements of  $\mathbb{R}_F$  which is  $\mathbb{Z}$ -valued.

Let  $F$  be a  $\mathbb{Z}$ -valued finite sequence of elements of  $\mathbb{R}_F$ . One can check that  $\sum F$  is integer.

Let  $f$  be a  $\mathbb{Z}$ -valued sequence of  $\mathbb{R}_F$ . Let us note that  $-f$  is  $\mathbb{Z}$ -valued.

Let  $g$  be a  $\mathbb{Z}$ -valued sequence of  $\mathbb{R}_F$ . Observe that  $f + g$  is  $\mathbb{Z}$ -valued and  $f - g$  is  $\mathbb{Z}$ -valued and  $f * g$  is  $\mathbb{Z}$ -valued.

Now we state the proposition:

- (39) Let us consider a non degenerated, non empty double loop structure  $L$ , and an element  $z$  of  $L$ . Then  $\text{LC}\langle z, 1_L \rangle = 1_L$ .

Let  $L$  be a non degenerated, non empty double loop structure and  $z$  be an element of  $L$ . One can check that  $\langle z, 1_L \rangle$  is monic.

Now we state the proposition:

- (40) Let us consider a non degenerated, non empty double loop structure  $L$ , and elements  $z_1, z_2$  of  $L$ . Then  $\text{LC}\langle z_1, z_2, 1_L \rangle = 1_L$ . The theorem is a consequence of (28) and (25).

Let  $L$  be a non degenerated, non empty double loop structure and  $z_1, z_2$  be elements of  $L$ . Let us observe that  $\langle z_1, z_2, 1_L \rangle$  is monic.

Let  $p$  be a  $\mathbb{Z}$ -valued polynomial over  $\mathbb{R}_F$ . Let us note that  $\text{LC} p$  is integer.

Now we state the proposition:

- (41) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure  $L$ , and a polynomial  $p$  over  $L$ . Then  $\deg(-p) = \deg p$ .

Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure  $L$  and polynomials  $p, q$  over  $L$ . Now we state the propositions:

- (42) If  $\deg p > \deg q$ , then  $\deg(p + q) = \deg p$ .  
 (43) If  $\deg p > \deg q$ , then  $\deg(p - q) = \deg p$ .  
 (44) If  $\deg p < \deg q$ , then  $\deg(p - q) = \deg q$ .  
 (45) Let us consider an add-associative, right zeroed, right complementable, distributive, non degenerated double loop structure  $L$ , and a polynomial  $p$  over  $L$ . Then  $\text{LC } p = -\text{LC}(-p)$ .  
 (46) Let us consider an add-associative, right zeroed, right complementable, associative, commutative, well unital, almost left invertible, distributive, non degenerated double loop structure  $L$ , and polynomials  $p, q$  over  $L$ . Then  $\text{LC}(p * q) = \text{LC } p \cdot \text{LC } q$ . The theorem is a consequence of (19).

Let us consider an add-associative, right zeroed, right complementable, distributive, non degenerated double loop structure  $L$ , a monic polynomial  $p$  over  $L$ , and a polynomial  $q$  over  $L$ . Now we state the propositions:

- (47) If  $\deg p > \deg q$ , then  $p + q$  is monic. The theorem is a consequence of (42).  
 (48) If  $\deg p > \deg q$ , then  $p - q$  is monic. The theorem is a consequence of (43).

Let  $L$  be an add-associative, right zeroed, right complementable, associative, commutative, well unital, almost left invertible, distributive, non degenerated double loop structure and  $p, q$  be monic polynomials over  $L$ . Let us note that  $p * q$  is monic.

Now we state the propositions:

- (49) Let us consider an Abelian, add-associative, right zeroed, right complementable, unital, distributive, non empty double loop structure  $L$ , elements  $z_1, z_2$  of  $L$ , and a polynomial  $p$  over  $L$ . Suppose  $\text{eval}(p, z_1) = z_2$ . Then  $\text{eval}(p - \langle z_2 \rangle, z_1) = 0_L$ .  
 (50) RATIONAL ROOT THEOREM:

Let us consider a  $\mathbb{Z}$ -valued polynomial  $p$  over  $\mathbb{R}_F$ , and an element  $e$  of  $\mathbb{R}_F$ . Suppose  $e$  is a root of  $p$ . Let us consider integers  $k, l$ . Suppose  $l \neq 0$  and  $e = \frac{k}{l}$  and  $k$  and  $l$  are relatively prime. Then

- (i)  $k \mid p(0)$ , and

(ii)  $l \mid LCp$ .

The theorem is a consequence of (7), (6), and (4).

(51) INTEGRAL ROOT THEOREM:

Let us consider a monic,  $\mathbb{Z}$ -valued polynomial  $p$  over  $\mathbb{R}_F$ , and a rational element  $e$  of  $\mathbb{R}_F$ . If  $e$  is a root of  $p$ , then  $e$  is integer. The theorem is a consequence of (50).

(52) Suppose  $1 \leq n$  and  $e = 2 \cdot \cos t$ . Then there exists a monic,  $\mathbb{Z}$ -valued polynomial  $p$  over  $\mathbb{R}_F$  such that

(i)  $\text{eval}(p, e) = 2 \cdot \cos(n \cdot t)$ , and

(ii)  $\text{deg } p = n$ , and

(iii) if  $n = 1$ , then  $p = \langle 0_{\mathbb{R}_F}, 1_{\mathbb{R}_F} \rangle$ , and

(iv) if  $n = 2$ , then there exists an element  $r$  of  $\mathbb{R}_F$  such that  $r = -2$  and  $p = \langle r, 0_{\mathbb{R}_F}, 1_{\mathbb{R}_F} \rangle$ .

PROOF: Define  $\mathcal{P}[\text{natural number}] \equiv$  if  $1 \leq \$_1$ , then there exists a monic,  $\mathbb{Z}$ -valued polynomial  $p$  over  $\mathbb{R}_F$  such that  $\text{eval}(p, e) = 2 \cdot \cos(\$_1 \cdot t)$  and  $\text{deg } p = \$_1$  and if  $\$_1 = 1$ , then  $p = \langle 0_{\mathbb{R}_F}, 1_{\mathbb{R}_F} \rangle$  and if  $\$_1 = 2$ , then there exists an element  $r$  of  $\mathbb{R}_F$  such that  $r = -2$  and  $p = \langle r, 0_{\mathbb{R}_F}, 1_{\mathbb{R}_F} \rangle$ .  $\mathcal{P}[1]$  by [11, (48), (40)].  $\mathcal{P}[2]$  by [6, (7)], (38), (28). For every non zero natural number  $k$  such that  $\mathcal{P}[k]$  and  $\mathcal{P}[k + 1]$  holds  $\mathcal{P}[k + 2]$  by [1, (13)], [13, (38)], (48), [10, (24)]. For every non zero natural number  $k$ ,  $\mathcal{P}[k]$  from [7, Sch. 1].  $\square$

(53) If  $0 \leq r \leq \frac{\pi}{2}$  and  $\frac{r}{\pi}$  is rational and  $\cos r$  is rational, then  $r \in \{0, \frac{\pi}{3}, \frac{\pi}{2}\}$ . The theorem is a consequence of (52), (1), (49), (48), (51), (3), and (18).

(54) Suppose  $2 \cdot \pi \cdot i \leq r \leq \frac{\pi}{2} + 2 \cdot \pi \cdot i$  and  $\frac{r}{\pi}$  is rational and  $\cos r$  is rational. Then  $r \in \{2 \cdot \pi \cdot i, \frac{\pi}{3} + 2 \cdot \pi \cdot i, \frac{\pi}{2} + 2 \cdot \pi \cdot i\}$ . The theorem is a consequence of (53).

(55) If  $\frac{\pi}{2} \leq r \leq \pi$  and  $\frac{r}{\pi}$  is rational and  $\cos r$  is rational, then  $r \in \{\frac{\pi}{2}, \frac{2\pi}{3}, \pi\}$ . The theorem is a consequence of (53).

(56) Suppose  $\frac{\pi}{2} + 2 \cdot \pi \cdot i \leq r \leq \pi + 2 \cdot \pi \cdot i$  and  $\frac{r}{\pi}$  is rational and  $\cos r$  is rational. Then  $r \in \{\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{2\pi}{3} + 2 \cdot \pi \cdot i, \pi + 2 \cdot \pi \cdot i\}$ . The theorem is a consequence of (55).

(57) Suppose  $\pi \leq r \leq \frac{3\pi}{2}$  and  $\frac{r}{\pi}$  is rational and  $\cos r$  is rational. Then  $r \in \{\pi, \frac{4\pi}{3}, \frac{3\pi}{2}\}$ . The theorem is a consequence of (53).

(58) Suppose  $\pi + 2 \cdot \pi \cdot i \leq r \leq \frac{3\pi}{2} + 2 \cdot \pi \cdot i$  and  $\frac{r}{\pi}$  is rational and  $\cos r$  is rational. Then  $r \in \{\pi + 2 \cdot \pi \cdot i, \frac{4\pi}{3} + 2 \cdot \pi \cdot i, \frac{3\pi}{2} + 2 \cdot \pi \cdot i\}$ . The theorem is a consequence of (57).

(59) Suppose  $\frac{3\pi}{2} \leq r \leq 2 \cdot \pi$  and  $\frac{r}{\pi}$  is rational and  $\cos r$  is rational. Then  $r \in \{\frac{3\pi}{2}, \frac{5\pi}{3}, 2 \cdot \pi\}$ . The theorem is a consequence of (53).

- (60) Suppose  $\frac{3\pi}{2} + 2 \cdot \pi \cdot i \leq r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$  and  $\frac{r}{\pi}$  is rational and  $\cos r$  is rational. Then  $r \in \{\frac{3\pi}{2} + 2 \cdot \pi \cdot i, \frac{5\pi}{3} + 2 \cdot \pi \cdot i, 2 \cdot \pi + 2 \cdot \pi \cdot i\}$ . The theorem is a consequence of (59).
- (61) If  $\frac{r}{\pi}$  is rational and  $\cos r$  is rational, then  $\cos r \in \{0, 1, -1, \frac{1}{2}, -\frac{1}{2}\}$ .
- (62) If  $0 \leq r \leq \frac{\pi}{2}$  and  $\frac{r}{\pi}$  is rational and  $\sin r$  is rational, then  $r \in \{0, \frac{\pi}{6}, \frac{\pi}{2}\}$ . The theorem is a consequence of (53).
- (63) Suppose  $2 \cdot \pi \cdot i \leq r \leq \frac{\pi}{2} + 2 \cdot \pi \cdot i$  and  $\frac{r}{\pi}$  is rational and  $\sin r$  is rational. Then  $r \in \{2 \cdot \pi \cdot i, \frac{\pi}{6} + 2 \cdot \pi \cdot i, \frac{\pi}{2} + 2 \cdot \pi \cdot i\}$ . The theorem is a consequence of (62).
- (64) If  $\frac{\pi}{2} \leq r \leq \pi$  and  $\frac{r}{\pi}$  is rational and  $\sin r$  is rational, then  $r \in \{\frac{\pi}{2}, \frac{5\pi}{6}, \pi\}$ . The theorem is a consequence of (62).
- (65) Suppose  $\frac{\pi}{2} + 2 \cdot \pi \cdot i \leq r \leq \pi + 2 \cdot \pi \cdot i$  and  $\frac{r}{\pi}$  is rational and  $\sin r$  is rational. Then  $r \in \{\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{5\pi}{6} + 2 \cdot \pi \cdot i, \pi + 2 \cdot \pi \cdot i\}$ . The theorem is a consequence of (64).
- (66) Suppose  $\pi \leq r \leq \frac{3\pi}{2}$  and  $\frac{r}{\pi}$  is rational and  $\sin r$  is rational. Then  $r \in \{\pi, \frac{7\pi}{6}, \frac{3\pi}{2}\}$ . The theorem is a consequence of (62).
- (67) Suppose  $\pi + 2 \cdot \pi \cdot i \leq r \leq \frac{3\pi}{2} + 2 \cdot \pi \cdot i$  and  $\frac{r}{\pi}$  is rational and  $\sin r$  is rational. Then  $r \in \{\pi + 2 \cdot \pi \cdot i, \frac{7\pi}{6} + 2 \cdot \pi \cdot i, \frac{3\pi}{2} + 2 \cdot \pi \cdot i\}$ . The theorem is a consequence of (66).
- (68) Suppose  $\frac{3\pi}{2} \leq r \leq 2 \cdot \pi$  and  $\frac{r}{\pi}$  is rational and  $\sin r$  is rational. Then  $r \in \{\frac{3\pi}{2}, \frac{11\pi}{6}, 2 \cdot \pi\}$ . The theorem is a consequence of (62).
- (69) Suppose  $\frac{3\pi}{2} + 2 \cdot \pi \cdot i \leq r \leq 2 \cdot \pi + 2 \cdot \pi \cdot i$  and  $\frac{r}{\pi}$  is rational and  $\sin r$  is rational. Then  $r \in \{\frac{3\pi}{2} + 2 \cdot \pi \cdot i, \frac{11\pi}{6} + 2 \cdot \pi \cdot i, 2 \cdot \pi + 2 \cdot \pi \cdot i\}$ . The theorem is a consequence of (68).
- (70) NIVEN'S THEOREM:  
If  $\frac{r}{\pi}$  is rational and  $\sin r$  is rational, then  $\sin r \in \{0, 1, -1, \frac{1}{2}, -\frac{1}{2}\}$ .

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