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Separability of Real Normed Spaces and Its Basic Properties

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Summary. In this article, the separability of real normed spaces and its properties are mainly formalized. In the first section, it is proved that a real normed subspace is separable if it is generated by a countable subset. We used here the fact that the rational numbers form a dense subset of the real numbers. In the second section, the basic properties of the separable normed spaces are discussed. It is applied to isomorphic spaces via bounded linear operators and double dual spaces. In the last section, it is proved that the completeness and reflexivity are transferred to sublinear normed spaces. The formalization is based on [34], and also referred to [7], [14] and [16].

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The notation and terminology used in this paper have been introduced in the following articles: [2], [4], [8], [26], [20], [21], [13], [9], [10], [22], [1], [25], [24], [15], [19], [6], [11], [23], [17], [32], [33], [27], [28], [29], [30], [31], [18], and [12].

1. Separability of Real Normed Space

Let X be a real linear space and A be a subset of X. The functor $\mathrm{Sums}_{\mathbb{Q}}\, A$ yielding a subset of X is defined by the term

(Def. 1) $\{\sum l$, where l is a linear combination of $A : \mathrm{rng}\, l \subseteq \mathbb{Q}\}$.

Let us consider a real normed space V and a real normed subspace V_1 of V. Now we state the propositions:

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(1) TopSpaceNorm V_1 is a subspace of TopSpaceNorm V.
PROOF: For every points x, y of MetricSpaceNorm V_1, (the distance of MetricSpaceNorm V_1)(x, y) = (the distance of MetricSpaceNorm V)(x, y) by [28, (16)], [19, (28)]. $\square$

(2) LinearTopSpaceNorm V_1 is a subspace of LinearTopSpaceNorm V. The theorem is a consequence of (1).

Now we state the proposition:

(3) Let us consider a real normed space X, and real normed subspaces Y, Z of X. Suppose there exists a subset A of X such that $A =$ the carrier of Y and $\overline{A} =$ the carrier of Z. Let us consider a subset D_0 of Y, and a subset D of Z. If D_0 is dense and $D_0 = D$, then D is dense.
PROOF: LinearTopSpaceNorm Z is a subspace of LinearTopSpaceNorm X and LinearTopSpaceNorm Y is a subspace of LinearTopSpaceNorm X. For every subset S of Z such that $S \neq \emptyset$ and S is open holds D meets S by [15, (16), (20)], [19, (5), (17), (4)]. $\square$

Let us consider an additive loop structure X and subsets A, B of X. Now we state the propositions:

(4) There exists a function F from $A + B$ into $A \times B$ such that F is one-to-one.
PROOF: Set $D = A + B$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exist points a, b of X such that $\$_1 = a + b$ and $a \in A$ and $b \in B$ and $\$_2 = \langle a, b \rangle$. For every object x such that $x \in D$ there exists an object y such that $y \in A \times B$ and $\mathcal{P}[x, y]$ by [12, (87)]. Consider F being a function from D into $A \times B$ such that for every object x such that $x \in D$ holds $\mathcal{P}[x, F(x)]$ from [10, Sch. 1]. For every objects x_1, x_2 such that $x_1, x_2 \in \operatorname{dom} F$ and $F(x_1) = F(x_2)$ holds $x_1 = x_2$. $\square$

(5) If A is countable and B is countable, then $A + B$ is countable. The theorem is a consequence of (4).

Now we state the proposition:

(6) Let us consider a non empty additive loop structure X, subsets A, B of X, a linear combination l_1 of A, and a linear combination l_2 of B. Suppose A misses B. Then there exists a linear combination l of $A \cup B$ such that

(i) the support of l = (the support of l_1) $\cup$ (the support of l_2), and

(ii) $l = l_1 + l_2$.

PROOF: Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ if $\$_1 \in$ the support of l_1, then $\$_2 = l_1(\$_1)$ and if $\$_1 \in$ the support of l_2, then $\$_2 = l_2(\$_1)$ and if $\$_1 \notin$ the support of l_1 and $\$_1 \notin$ the support of l_2, then $\$_2 = 0$. Consider l being a function from the carrier of X into $\mathbb{R}$ such that for every object x such that $x \in$ the carrier of X holds $\mathcal{P}[x, l(x)]$ from [10, Sch. 1]. Reconsider $T =$ (the support of l_1) $\cup$ (the support of l_2) as a finite subset of X. For every element x of X such that $x \notin T$ holds $l(x) = 0$. For every element v of X, $l(v) = l_1(v) + l_2(v)$. □

Let us consider a non empty additive loop structure X, subsets A, B of X, and a linear combination l of $A \cup B$. Now we state the propositions:

(7) There exists a linear combination l_1 of A such that

(i) the support of $l_1 =$ (the support of l) $\setminus B$, and

(ii) for every element x of X such that $x \in$ the support of l_1 holds $l_1(x) = l(x)$.

PROOF: Reconsider $T_1 =$ (the support of l) $\setminus B$ as a finite subset of X. Define $\mathcal{Q}[\text{object}, \text{object}] \equiv$ if $\$_1 \in T_1$, then $\$_2 = l(\$_1)$ and if $\$_1 \notin T_1$, then $\$_2 = 0$. Consider l_1 being a function from the carrier of X into $\mathbb{R}$ such that for every object x such that $x \in$ the carrier of X holds $\mathcal{Q}[x, l_1(x)]$ from [10, Sch. 1]. □

(8) Suppose A misses B. Then there exists a linear combination l_1 of A and there exists a linear combination l_2 of B such that the support of $l =$ (the support of l_1)$\cup$(the support of l_2) and $l = l_1 + l_2$ and the support of $l_1 =$ (the support of l)$\setminus B$ and the support of $l_2 =$ (the support of l)$\setminus A$. The theorem is a consequence of (7).

Now we state the propositions:

(9) Let us consider a real linear space X, subsets A, B of X, a linear combination l_1 of A, and a linear combination l_2 of B. Suppose $\operatorname{rng} l_1 \subseteq \mathbb{Q}$ and $\operatorname{rng} l_2 \subseteq \mathbb{Q}$ and A misses B. Then there exists a linear combination l of $A \cup B$ such that

(i) the support of $l =$ (the support of l_1) $\cup$ (the support of l_2), and

(ii) $\operatorname{rng} l \subseteq \mathbb{Q}$, and

(iii) $\sum l = \sum l_1 + \sum l_2$.

The theorem is a consequence of (6).

(10) Let us consider a real linear space X, subsets A, B of X, and a linear combination l of $A \cup B$. Suppose $\operatorname{rng} l \subseteq \mathbb{Q}$ and A misses B. Then there exists a linear combination l_1 of A and there exists a linear combination l_2 of B such that $\operatorname{rng} l_1 \subseteq \mathbb{Q}$ and $\operatorname{rng} l_2 \subseteq \mathbb{Q}$ and $\sum l = \sum l_1 + \sum l_2$. The theorem is a consequence of (8).

(11) Let us consider a real linear space X, and finite subsets A, B of X. Suppose A misses B. Then $\operatorname{Sums}_{\mathbb{Q}} A + \operatorname{Sums}_{\mathbb{Q}} B = \operatorname{Sums}_{\mathbb{Q}}(A \cup B)$. The theorem is a consequence of (9) and (10).

Let X be a real linear space and A be a finite subset of X. Observe that $\mathrm{Sums}_{\mathbb{Q}}\, A$ is countable.

Now we state the proposition:

(12) Let us consider a real linear space X, a sequence x of X, and a finite subset A of X. Suppose $A \subseteq \operatorname{rng} x$. Then there exists a natural number n such that $A \subseteq \operatorname{rng}(x \restriction \mathbb{Z}_n)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite subset A of X such that $\overline{\overline{A}} = \$_1$ and $A \subseteq \operatorname{rng} x$ there exists a natural number n such that $A \subseteq \operatorname{rng}(x \restriction \mathbb{Z}_n)$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (44)], [12, (31)], [3, (42)], [10, (11)]. For every natural number k, $\mathcal{P}[k]$ from [5, Sch. 2]. $\square$

Let X be a real linear space and x be a sequence of X. One can verify that $\mathrm{Sums}_{\mathbb{Q}} \operatorname{rng} x$ is countable.

Now we state the propositions:

(13) Let us consider a real normed space X, and a sequence x of X. Then $\mathrm{Sums}_{\mathbb{Q}} \operatorname{rng} x$ is a subset of the carrier of $\mathrm{NLin} \operatorname{rng} x$.

PROOF: Set $D = \mathrm{Sums}_{\mathbb{Q}} \operatorname{rng} x$. For every object z such that $z \in D$ holds $z \in$ the carrier of $\mathrm{NLin} \operatorname{rng} x$ by [30, (14)]. $\square$

(14) Let us consider a real normed space X, and a subset Y of X. Then

(i) the carrier of $\mathrm{NLin}\, Y \subseteq$ the carrier of $\mathrm{ClNLin}(Y)$, and

(ii) there exists a subset Z of X such that $Z =$ the carrier of $\mathrm{NLin}\, Y$ and $\overline{Z} =$ the carrier of $\mathrm{ClNLin}(Y)$.

(15) Let us consider a real normed space X, and a sequence x of X. Then $\mathrm{Sums}_{\mathbb{Q}} \operatorname{rng} x$ is a countable subset of the carrier of $\mathrm{ClNLin}(\operatorname{rng} x)$. The theorem is a consequence of (13) and (14).

(16) Let us consider real numbers z, e. Suppose $0 < e$. Then there exists an element q of $\mathbb{Q}$ such that

(i) $q \neq 0$, and

(ii) $|z - q| < e$.

(17) Let us consider a real normed space X, a point w of X, a real number e, and a linear combination l of $\{w\}$. Suppose $0 < e$. Then there exists a linear combination m of $\{w\}$ such that

(i) the support of $m =$ the support of l, and

(ii) $\operatorname{rng} m \subseteq \mathbb{Q}$, and

(iii) $\|\sum l - \sum m\| < e$.

The theorem is a consequence of (16).

(18) Let us consider a real normed space X, a subset A of X, a real number e, and a linear combination l of A. Suppose $0 < e$. Then there exists a linear combination m of A such that

(i) the support of $m =$ the support of l, and

(ii) $\operatorname{rng} m \subseteq \mathbb{Q}$, and

(iii) $\|\sum l - \sum m\| < e$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every real number e for every linear combination l of A such that $0 < e$ and $\overline{\overline{\text{the support of } l}} = \$_1$ there exists a linear combination m of A such that the support of $m =$ the support of l and $\operatorname{rng} m \subseteq \mathbb{Q}$ and $\|\sum l - \sum m\| < e$. $\mathcal{P}[0]$ by [29, (34), (44), (42)], [30, (2)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (44)], [12, (31)], [3, (42)], (8). For every natural number k, $\mathcal{P}[k]$ from [5, Sch. 2]. □

Let us consider a real normed space X and a sequence x of X. Now we state the propositions:

(19) $\mathrm{Sums}_{\mathbb{Q}} \operatorname{rng} x$ is a dense subset of the carrier of $\mathrm{NLin} \operatorname{rng} x$.

(20) $\mathrm{Sums}_{\mathbb{Q}} \operatorname{rng} x$ is a dense subset of the carrier of $\mathrm{ClNLin}(\operatorname{rng} x)$.

Now we state the proposition:

(21) Let us consider a real normed space X. Suppose there exists a subset D of the carrier of X such that D is dense and countable. Then X is separable.

2. Basic Properties of Separable Spaces

Let X be a real normed space and x be a sequence of X. Let us observe that $\mathrm{ClNLin}(\operatorname{rng} x)$ is separable.

Now we state the propositions:

(22) Let us consider a real normed space X, a real normed subspace Y of X, and a Lipschitzian linear functional L in X. Then $L\restriction$(the carrier of Y) is a Lipschitzian linear functional in Y.

PROOF: Set $Y_1 =$ the carrier of Y. Reconsider $L_1 = L\restriction Y_1$ as a functional in Y. L_1 is additive by [9, (49)], [19, (28)]. L_1 is homogeneous by [9, (49)], [19, (28)]. Consider K being a real number such that $0 \leqslant K$ and for every point x of X, $|L(x)| \leqslant K \cdot \|x\|$. For every point x of Y, $|L_1(x)| \leqslant K \cdot \|x\|$ by [19, (28)], [9, (49)]. □

(23) Let us consider real normed spaces X, Y, a subset A of X, a subset B of Y, and a Lipschitzian linear operator L from X into Y. Suppose L is isomorphism and $B = L^{\circ}A$. Then A is dense if and only if B is dense.

(24) Let us consider real normed spaces X, Y. Suppose there exists a Lipschitzian linear operator L from X into Y such that L is isomorphism. Then X is separable if and only if Y is separable. The theorem is a consequence of (23).

(25) Let us consider a real normed space X. Suppose X is non trivial and reflexive. Then X is separable if and only if DualSp(DualSp(X)) is separable. The theorem is a consequence of (24).

3. Completeness and Reflexivity of Sublinear Normed Spaces

Now we state the proposition:

(26) Let us consider a real normed space X, and subsets Y, Z of X. Suppose $Z =$ the carrier of Lin(Y). Then the carrier of Lin(Z) $= Z$.

Let us consider a real Banach space X and a subset Y of X. Now we state the propositions:

(27) There exists a subset Z of X such that

- (i) $Z =$ the carrier of Lin(Y), and
- (ii) ClNLin(Y) = NLin $\overline{Z}$, and
- (iii) $\overline{Z}$ is linearly closed, and
- (iv) $\overline{Z} \neq \emptyset$.

(28) ClNLin(Y) is a real Banach space. The theorem is a consequence of (27).

(29) If X is reflexive, then ClNLin(Y) is reflexive. The theorem is a consequence of (27).

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