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Bidual Spaces and Reflexivity of Real Normed Spaces[1]

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Summary. In this article, we considered bidual spaces and reflexivity of real normed spaces. At first we proved some corollaries applying Hahn-Banach theorem and showed related theorems. In the second section, we proved the norm of dual spaces and defined the natural mapping, from real normed spaces to bidual spaces. We also proved some properties of this mapping. Next, we defined real normed space of $\mathbb{R}$, real number spaces as real normed spaces and proved related theorems. We can regard linear functionals as linear operators by this definition. Accordingly we proved Uniform Boundedness Theorem for linear functionals using the theorem (5) from [21]. Finally, we defined reflexivity of real normed spaces and proved some theorems about isomorphism of linear operators. Using them, we proved some properties about reflexivity. These formalizations are based on [19], [20], [8] and [1].

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The notation and terminology used in this paper have been introduced in the following articles: [2], [14], [7], [3], [4], [16], [22], [24], [15], [18], [13], [5], [10], [29], [25], [26], [11], [28], [12], and [6].

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1. The Application of Hahn-Banach Theorem

Now we state the propositions:

(1) Let us consider a real normed space V, a real normed subspace X of V, a point x_0 of V, and a real number d. Suppose there exists a non empty subset Z of $\mathbb{R}$ such that $Z = \{\|x - x_0\|$, where x is a point of $V : x \in X\}$ and $d = \inf Z > 0$. Then

(i) $x_0 \notin X$, and

(ii) there exists a point G of DualSp(V) such that for every point x of V such that $x \in X$ holds (Bound2Lipschitz(G, V))(x) $= 0$ and (Bound2Lipschitz(G, V))(x_0) $= 1$ and $\|G\| = \frac{1}{d}$.

Proof: Consider Z being a non empty subset of $\mathbb{R}$ such that $Z = \{\|x - x_0\|$, where x is a point of $V : x \in X\}$ and $d = \inf Z > 0$. Set $M_0 = \{z + a \cdot x_0$, where z is a point of V, a is a real number : $z \in X\}$. Set $M =$ NLin M_0. M_0 is linearly closed by [25, (20), (21)]. For every point v of M, there exists a point x of V and there exists a real number a such that $v = x + a \cdot x_0$ and $x \in X$ by [13, (31)]. Reconsider $r_0 = 0$ as a real number. For every extended real r such that $r \in Z$ holds $r_0 \leqslant r$. For every points x_1, x_2 of V and for every real numbers a_1, a_2 such that x_1, $x_2 \in X$ and $x_1 + a_1 \cdot x_0 = x_2 + a_2 \cdot x_0$ holds $x_1 = x_2$ and $a_1 = a_2$ by [26, (5), (35), (15)]. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ there exists a point z of V and there exists a real number a such that $z \in X$ and $\$_1 = z + a \cdot x_0$ and $\$_2 = a$. For every element v of M, there exists an element a of $\mathbb{R}$ such that $\mathcal{P}[v, a]$. Consider f being a function from M into $\mathbb{R}$ such that for every element x of M, $\mathcal{P}[x, f(x)]$ from [4, Sch. 3]. For every point v of M and for every point z of V and for every real number a such that $z \in X$ and $v = z + a \cdot x_0$ holds $f(v) = a$. f is a linear functional in M by [13, (28)], [25, (20), (21)]. For every point v of M, $|f(v)| \leqslant \frac{1}{d} \cdot \|v\|$ by [17, (2)], [18, (2)], [26, (30), (25)]. Reconsider $F = f$ as a point of DualSp(M). Consider g being a Lipschitzian linear functional in V, G being a point of DualSp(V) such that $g = G$ and $g{\restriction}$(the carrier of M) $= f$ and $\|G\| = \|F\|$. For every point x of V such that $x \in X$ holds (Bound2Lipschitz(G, V))(x) $= 0$ by [26, (10)], [3, (49)]. □

(2) Let us consider a real normed space V, a non empty subset Y of V, and a point x_0 of V. Suppose Y is linearly closed and closed and $x_0 \notin Y$. Then there exists a point G of DualSp(V) such that

(i) for every point x of V such that $x \in Y$ holds (Bound2Lipschitz(G, V))(x) $= 0$, and

(ii) $(\mathrm{Bound2Lipschitz}(G, V))(x_0) = 1$.

Proof: Set $X = \mathrm{NLin}\, Y$. Set $Z = \{\|x - x_0\|$, where x is a point of $V : x \in X\}$. Reconsider $r_0 = 0$ as a real number. For every extended real r such that $r \in Z$ holds $r_0 \leqslant r$. Reconsider $d = \inf Z$ as a real number. $d > 0$ by [9, (16), (7)], [18, (7)]. Consider G being a point of $\mathrm{DualSp}(V)$ such that for every point x of V such that $x \in X$ holds $(\mathrm{Bound2Lipschitz}(G, V))(x) = 0$ and $(\mathrm{Bound2Lipschitz}(G, V))(x_0) = 1$ and $\|G\| = \frac{1}{d}$. □

Let us consider a real normed space V and a point x_0 of V.
Let us assume that $x_0 \neq 0_V$. Now we state the propositions:

(3) There exists a point G of $\mathrm{DualSp}(V)$ such that

(i) $(\mathrm{Bound2Lipschitz}(G, V))(x_0) = 1$, and

(ii) $\|G\| = \frac{1}{\|x_0\|}$.

Proof: Set $X = \mathrm{NLin}\{0_V\}$. Set $Y =$ the carrier of $\mathrm{Lin}(\{0_V\})$. For every object s, $s \in Y$ iff $s \in \{0_V\}$ by [27, (8)]. Set $Z = \{\|x - x_0\|$, where x is a point of $V : x \in X\}$. For every object s, $s \in Z$ iff $s \in \{\|x_0\|\}$ by [18, (2)]. Reconsider $d = \inf Z$ as a real number. Consider G being a point of $\mathrm{DualSp}(V)$ such that for every point x of V such that $x \in X$ holds $(\mathrm{Bound2Lipschitz}(G, V))(x) = 0$ and $(\mathrm{Bound2Lipschitz}(G, V))(x_0) = 1$ and $\|G\| = \frac{1}{d}$. □

(4) There exists a point F of $\mathrm{DualSp}(V)$ such that

(i) $\|F\| = 1$, and

(ii) $(\mathrm{Bound2Lipschitz}(F, V))(x_0) = \|x_0\|$.

The theorem is a consequence of (3).

Let us consider a real normed space V.
Let us assume that V is not trivial. Now we state the propositions:

(5) There exists a point F of $\mathrm{DualSp}(V)$ such that $\|F\| = 1$. The theorem is a consequence of (4).

(6) $\mathrm{DualSp}(V)$ is not trivial. The theorem is a consequence of (5).

2. Bidual Spaces of Real Normed Spaces

Let us consider a real normed space V and a point x of V. Now we state the propositions:

(7) Suppose V is not trivial. Then

(i) there exists a non empty subset X of $\mathbb{R}$ such that $X = \{|(\mathrm{Bound2Lipschitz}(F, V))(x)|$, where F is a point of $\mathrm{DualSp}(V) : \|F\| = 1\}$ and $\|x\| = \sup X$, and

(ii) there exists a non empty subset Y of $\mathbb{R}$ such that $Y = \{|(\text{Bound2Lipschitz}(F, V))(x)|$, where F is a point of DualSp(V) : $\|F\| \leqslant 1\}$ and $\|x\| = \sup Y$.

The theorem is a consequence of (5) and (4).

(8) If for every Lipschitzian linear functional f in V, $f(x) = 0$, then $x = 0_V$. The theorem is a consequence of (3).

Let X be a real normed space and x be a point of X. The functor Bidual x yielding a point of DualSp(DualSp(X)) is defined by

(Def. 1) for every point f of DualSp(X), $it(f) = f(x)$.

The functor BidualFunc X yielding a function from X into DualSp(DualSp(X)) is defined by

(Def. 2) for every point x of X, $it(x) =$ Bidual x.

Let us observe that BidualFunc X is additive and homogeneous and BidualFunc X is one-to-one.

Let us consider a real normed space X.

Let us assume that X is not trivial. Now we state the propositions:

(9) (i) BidualFunc X is a linear operator from X into DualSp(DualSp(X)), and

(ii) for every point x of X, $\|x\| = \|(\text{BidualFunc}\, X)(x)\|$.

(10) There exists a real normed subspace D of DualSp(DualSp(X)) and there exists a Lipschitzian linear operator L from X into D such that L is bijective and $D = \Im(\text{BidualFunc}\, X)$ and for every point x of X, $L(x) =$ Bidual x and for every point x of X, $\|x\| = \|L(x)\|$.

PROOF: Set $F =$ BidualFunc X. Set $V_1 = \text{rng}\, F$. $V_1 \neq \emptyset$ by [29, (42)]. Reconsider $L =$ BidualFunc X as a function from X into $\Im(F)$. L is additive by [13, (28)]. L is homogeneous by [13, (28)]. For every point x of X, $\|x\| = \|L(x)\|$ by [13, (28)]. □

3. Uniform Boundedness Theorem for Linear Functionals

The real normed space of $\mathbb{R}$ yielding a real normed space is defined by the term

(Def. 3) $\langle \mathbb{R}, 0(\in \mathbb{R}), +_{\mathbb{R}}, \cdot_{\mathbb{R}}, |\square|_{\mathbb{R}} \rangle$.

Now we state the proposition:

(11) Let us consider a real normed space X, an element x of $\mathbb{R}$, and a point v of the real normed space of $\mathbb{R}$. If $x = v$, then $-x = -v$.

Let us consider a real normed space X and an object x. Now we state the propositions:

(12) x is an additive, homogeneous function from X into $\mathbb{R}$ if and only if x is an additive, homogeneous function from X into the real normed space of $\mathbb{R}$.

(13) x is a Lipschitzian, additive, homogeneous function from X into $\mathbb{R}$ if and only if x is a Lipschitzian, additive, homogeneous function from X into the real normed space of $\mathbb{R}$. The theorem is a consequence of (12).

Now we state the propositions:

(14) Let us consider a real normed space X. Then the carrier of $\mathrm{DualSp}(X) =$ the carrier of the real norm space of bounded linear operators from X into the real normed space of $\mathbb{R}$. The theorem is a consequence of (13).

(15) Let us consider a real normed space X, points x, y of $\mathrm{DualSp}(X)$, and points v, w of the real norm space of bounded linear operators from X into the real normed space of $\mathbb{R}$. If $x = v$ and $y = w$, then $x + y = v + w$.
Proof: Reconsider $z = x + y$ as a point of $\mathrm{DualSp}(X)$. Reconsider $u = v + w$ as a point of the real norm space of bounded linear operators from X into the real normed space of $\mathbb{R}$. For every object t such that $t \in \mathrm{dom}\, z$ holds $z(t) = u(t)$ by [14, (29)], [22, (35)]. □

(16) Let us consider a real normed space X, an element a of $\mathbb{R}$, a point x of $\mathrm{DualSp}(X)$, and a point v of the real norm space of bounded linear operators from X into the real normed space of $\mathbb{R}$. If $x = v$, then $a \cdot x = a \cdot v$.
Proof: Reconsider $z = a \cdot x$ as a point of $\mathrm{DualSp}(X)$. Reconsider $u = a \cdot v$ as a point of the real norm space of bounded linear operators from X into the real normed space of $\mathbb{R}$. For every object t such that $t \in \mathrm{dom}\, z$ holds $z(t) = u(t)$ by [14, (30)], [22, (36)]. □

Let us consider a real normed space X, a point x of $\mathrm{DualSp}(X)$, and a point v of the real norm space of bounded linear operators from X into the real normed space of $\mathbb{R}$.

Let us assume that $x = v$. Now we state the propositions:

(17) $-x = -v$. The theorem is a consequence of (16).

(18) $\|x\| = \|v\|$.

Now we state the propositions:

(19) Let us consider a real normed space X, and a subset L of X. Suppose X is not trivial and for every point f of $\mathrm{DualSp}(X)$, there exists a real number K_1 such that $0 \leqslant K_1$ and for every point x of X such that $x \in L$ holds $|f(x)| \leqslant K_1$. Then there exists a real number M such that

 (i) $0 \leqslant M$, and

(ii) for every point x of X such that $x \in L$ holds $\|x\| \leqslant M$.

The theorem is a consequence of (14) and (18).

(20) Let us consider a real normed space X, and a non empty subset L of X. Suppose X is not trivial and for every point f of DualSp(X), there exists a subset Y_1 of $\mathbb{R}$ such that $Y_1 = \{|f(x)|$, where x is a point of $X : x \in L\}$ and $\sup Y_1 < +\infty$. Then there exists a subset Y of $\mathbb{R}$ such that

(i) $Y = \{\|x\|$, where x is a point of $X : x \in L\}$, and

(ii) $\sup Y < +\infty$.

PROOF: For every point f of DualSp(X), there exists a real number K_1 such that $0 \leqslant K_1$ and for every point x of X such that $x \in L$ holds $|f(x)| \leqslant K_1$ by [2, (46)]. Consider M being a real number such that $0 \leqslant M$ and for every point x of X such that $x \in L$ holds $\|x\| \leqslant M$. Consider x_0 being an object such that $x_0 \in L$. Set $Y = \{\|x\|$, where x is a point of $X : x \in L\}$. $Y \subseteq \mathbb{R}$. For every extended real r such that $r \in Y$ holds $r \leqslant M$. □

4. Reflexivity of Real Normed Spaces

Let X be a real normed space. We say that X is reflexive if and only if

(Def. 4) BidualFunc X is onto.

Let us consider a real normed space X. Now we state the propositions:

(21) X is reflexive if and only if for every point f of DualSp(DualSp(X)), there exists a point x of X such that for every point g of DualSp(X), $f(g) = g(x)$.

(22) X is reflexive if and only if $\Im$(BidualFunc X) = DualSp(DualSp(X)).

(23) If X is non trivial and reflexive, then X is a real Banach space.

PROOF: For every sequence s_1 of X such that s_1 is Cauchy sequence by norm holds s_1 is convergent by [23, (8)], [3, (13)], [26, (16)], [4, (113)]. □

Now we state the propositions:

(24) Let us consider a real Banach space X, and a non empty subset M of X. Suppose X is reflexive and M is linearly closed and closed. Then NLin M is reflexive.

PROOF: Set M_0 = NLin M. For every point y of DualSp(DualSp(M_0)), there exists a point x of M_0 such that for every point g of DualSp(M_0), $y(g) = g(x)$ by [4, (32)], [13, (28)], [3, (49)], [14, (26), (29), (30)]. □

(25) Let us consider real normed spaces X, Y, a Lipschitzian linear operator L from X into Y, and a Lipschitzian linear functional y in Y. Then $y \cdot L$ is a Lipschitzian linear functional in X.

Proof: Consider M being a real number such that $0 \leqslant M$ and for every vector x of X, $\|L(x)\| \leqslant M \cdot \|x\|$. Set $x = y \cdot L$. For every vectors v, w of X, $x(v + w) = x(v) + x(w)$ by [3, (13)]. For every vector v of X and for every real number r, $x(r \cdot v) = r \cdot x(v)$ by [3, (13)]. Consider N being a real number such that $0 \leqslant N$ and for every vector v of Y, $|y(v)| \leqslant N \cdot \|v\|$. For every vector v of X, $|x(v)| \leqslant M \cdot N \cdot \|v\|$ by [3, (13)]. □

(26) Let us consider real normed spaces X, Y, and a Lipschitzian linear operator L from X into Y. Suppose L is isomorphism. Then there exists a Lipschitzian linear operator T from DualSp(X) into DualSp(Y) such that

(i) T is isomorphism, and

(ii) for every point x of DualSp(X), $T(x) = x \cdot L^{-1}$.

Proof: Consider K being a Lipschitzian linear operator from Y into X such that $K = L^{-1}$ and K is isomorphism. Define $\mathcal{P}[\text{function}, \text{function}] \equiv \$_2 = \$_1 \cdot K$. For every element x of DualSp(X), there exists an element y of DualSp(Y) such that $\mathcal{P}[x, y]$. Consider T being a function from DualSp(X) into DualSp(Y) such that for every element x of DualSp(X), $\mathcal{P}[x, T(x)]$ from [4, Sch. 3]. For every points v, w of DualSp(X), $T(v + w) = T(v) + T(w)$ by [3, (13)], [14, (29)]. For every point v of DualSp(X) and for every real number r, $T(r \cdot v) = r \cdot T(v)$ by [3, (13)], [14, (30)]. For every object v such that $v \in$ the carrier of DualSp(Y) there exists an object s such that $s \in$ the carrier of DualSp(X) and $v = T(s)$ by (25), [29, (36)], [3, (39)], [29, (51)]. For every point v of DualSp(X), $\|T(v)\| = \|v\|$ by [3, (34), (13)], [14, (23)]. For every objects x_1, x_2 such that $x_1, x_2 \in$ the carrier of DualSp(X) and $T(x_1) = T(x_2)$ holds $x_1 = x_2$ by [26, (16), (5)], [18, (6)]. □

(27) Let us consider real normed spaces X, Y, a Lipschitzian linear operator L from X into Y, and a Lipschitzian linear operator T from DualSp(X) into DualSp(Y). Suppose L is isomorphism and T is isomorphism and for every point x of DualSp(X), $T(x) = x \cdot L^{-1}$. Then there exists a Lipschitzian linear operator S from DualSp(Y) into DualSp(X) such that

(i) S is isomorphism, and

(ii) $S = T^{-1}$, and

(iii) for every point y of DualSp(Y), $S(y) = y \cdot L$.

Proof: Consider K being a Lipschitzian linear operator from Y into X such that $K = L^{-1}$ and K is isomorphism. Consider S being a Lipschitzian linear operator from DualSp(Y) into DualSp(X) such that S is isomorphism and for every point y of DualSp(Y), $S(y) = y \cdot K^{-1}$. For every objects $y, x, y \in$ the carrier of DualSp(Y) and $S(y) = x$ iff $x \in$ the carrier of DualSp(X) and $T(x) = y$ by [4, (5)], [29, (36)], [3, (39)], [29, (51)]. □

(28) Let us consider real normed spaces X, Y. Suppose there exists a Lipschitzian linear operator L from X into Y such that L is isomorphism. Then X is reflexive if and only if Y is reflexive.

(29) Let us consider a real normed space X. Suppose X is not trivial. Then there exists a Lipschitzian linear operator L from X into $\Im(\mathrm{BidualFunc}\, X)$ such that L is isomorphism. The theorem is a consequence of (10).

(30) Let us consider a real Banach space X. Suppose X is not trivial. Then X is reflexive if and only if DualSp(X) is reflexive.

PROOF: DualSp(X) is not trivial. Consider L being a Lipschitzian linear operator from X into $\Im(\mathrm{BidualFunc}\, X)$ such that L is isomorphism. Set $f = \mathrm{BidualFunc}\, X$. $\mathrm{rng}\, f \neq \emptyset$ by [29, (42)]. $\Im(f)$ is reflexive. □

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