Special Issue: 25 years of the Mizar Mathematical Library

FORMALIZED MATHEMATICS
Vol. 22, No. 2, Pages 157–166, 2014
DOI: 10.2478/forma-2014-0016

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Cauchy Mean Theorem

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Summary. The purpose of this paper was to prove formally, using the Mizar language, Arithmetic Mean/Geometric Mean theorem known maybe better under the name of AM-GM inequality or Cauchy mean theorem. It states that the arithmetic mean of a list of a non-negative real numbers is greater than or equal to the geometric mean of the same list.

The formalization was tempting for at least two reasons: one of them, perhaps the strongest, was that the proof of this theorem seemed to be relatively easy to formalize (e.g. the weaker variant of this was proven in [13]). Also Jensen's inequality is already present in the Mizar Mathematical Library. We were impressed by the beauty and elegance of the simple proof by induction and so we decided to follow this specific way.

The proof follows similar lines as that written in Isabelle [18]; the comparison of both could be really interesting as it seems that in both systems the number of lines needed to prove this are really close.

This theorem is item #38 from the "Formalizing 100 Theorems" list maintained by Freek Wiedijk at `http://www.cs.ru.nl/F.Wiedijk/100/` .

MSC: 26A06 26A12 03B35

Keywords: geometric mean; arithmetic mean; AM-GM inequality; Cauchy mean theorem

MML identifier: RVSUM_3, version: 8.1.03 5.23.1210

The notation and terminology used in this paper have been introduced in the following articles: [1], [2], [7], [5], [12], [20], [9], [6], [23], [21], [3], [17], [4], [15], [19], [14], [26], [16], [10], [22], [25], and [11].

ISSN 1426–2630(Print), 1898-9934(Online)

1. Preliminaries

Let us consider real numbers x, y, z, w. Now we state the propositions:

(1) If $|x-y|<|z-w|$, then $(x-y)^{2}<(z-w)^{2}$.

(2) If $|x-y|<|z-w|$ and $x+y=z+w$, then $x \cdot y > z \cdot w$. The theorem is a consequence of (1).

Let f be a real-valued finite sequence. We introduce f is positive as a synonym of f is positive yielding.

Observe that f is positive if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let us consider a natural number n. If $n \in \operatorname{dom} f$, then $f(n)>0$.

Note that there exists a real-valued finite sequence which is non empty, constant, and positive and there exists a real-valued finite sequence which is non empty, non constant, and positive.

Let f be a non empty real-valued finite sequence and n be a natural number. One can verify that $f \restriction \operatorname{Seg} n$ is real-valued.

Let f be a positive non empty real-valued finite sequence. Let us note that $f \restriction \operatorname{Seg} n$ is positive.

Let f be a finite sequence. We introduce f is homogeneous as a synonym of f is constant.

Let f be a finite sequence. We introduce f is heterogeneous as an antonym of f is homogeneous.

Let us consider real-valued finite sequences R_1, R_2. Now we state the propositions:

(3) Suppose $\operatorname{len} R_1 = \operatorname{len} R_2$ and for every natural number j such that $j \in \operatorname{Seg} \operatorname{len} R_1$ holds $R_1(j) \leqslant R_2(j)$ and there exists a natural number j such that $j \in \operatorname{Seg} \operatorname{len} R_1$ and $R_1(j) < R_2(j)$. Then $\sum R_1 < \sum R_2$.

(4) If R_1 and R_2 are fiberwise equipotent, then $\prod R_1 = \prod R_2$. Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite sequences f, g of elements of $\mathbb{R}$ such that f and g are fiberwise equipotent and $\operatorname{len} f = \$_1$ holds $\prod f = \prod g$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [2, (75)], [3, (13)], [24, (25)], [8, (10), (4), (5)]. $\mathcal{P}[0]$ by [16, (3)]. For every natural number n, $\mathcal{P}[n]$ from [3, Sch. 2]. $\square$

2. Arithmetic Mean and Geometric Mean

Let f be a real-valued finite sequence. The functor $\operatorname{Mean} f$ yielding a real number is defined by the term

(Def. 2) $\frac{\sum f}{\operatorname{len} f}$.

Let f be a positive real-valued finite sequence. The functor $\operatorname{GMean} f$ yielding a real number is defined by the term

(Def. 3) $\sqrt[\mathrm{len}\, f]{\prod f}$.

Let us consider a real-valued finite sequence f. Now we state the propositions:

(5) $\sum f = \mathrm{len}\, f \cdot \mathrm{Mean}\, f$.

(6) $\mathrm{Mean}(f \frown \langle \mathrm{Mean}\, f \rangle) = \mathrm{Mean}\, f$. The theorem is a consequence of (5).

Let f be a non empty constant real-valued finite sequence. Observe that the value of f is real.

Let us consider a non empty constant real-valued finite sequence f. Now we state the propositions:

(7) $\sum f = (\text{the value of } f) \cdot \mathrm{len}\, f$.

(8) $\prod f = (\text{the value of } f)^{\mathrm{len}\, f}$.

(9) $\mathrm{Mean}\, f =$ the value of f. The theorem is a consequence of (7).

Let us consider a non empty constant positive real-valued finite sequence f. Now we state the propositions:

(10) The value of $f > 0$.

(11) $\mathrm{GMean}\, f =$ the value of f. The theorem is a consequence of (10) and (8).

Let f be a non empty positive real-valued finite sequence. Observe that $\mathrm{Mean}\, f$ is positive.

Let us note that $\prod f$ is positive.

Let f be a positive non empty real-valued finite sequence. Note that $\mathrm{GMean}\, f$ is positive.

3. Heterogeneity of a Finite Sequence

Let f be a real-valued finite sequence. The functor $\mathrm{HetSet}\, f$ yielding a subset of $\mathbb{N}$ is defined by the term

(Def. 4) $\{n$, where n is a natural number : $n \in \mathrm{dom}\, f$ and $f(n) \neq \mathrm{Mean}\, f\}$.

One can verify that $\mathrm{HetSet}\, f$ is finite.

Let f be a positive non empty real-valued finite sequence. Let us observe that $\mathrm{HetSet}\, f$ is upper bounded lower bounded and real-membered.

Let f be a real-valued finite sequence. The functor $\mathrm{Het}\, f$ yielding a natural number is defined by the term

(Def. 5) $\overline{\overline{\mathrm{HetSet}\, f}}$.

Now we state the propositions:

(12) Let us consider a real-valued finite sequence f. If $\mathrm{Het}\, f = 0$, then f is homogeneous.

(13) Let us consider a non empty real-valued finite sequence f. If $\mathrm{Het}\, f \neq 0$, then f is heterogeneous. The theorem is a consequence of (9).

Let f be a heterogeneous positive non empty real-valued finite sequence. Note that $\operatorname{HetSet} f$ is non empty.

Now we state the proposition:

(14) Let us consider a non empty homogeneous positive real-valued finite sequence f. Then $\operatorname{Mean} f = \operatorname{GMean} f$. The theorem is a consequence of (9) and (11).

Let f_1, f_2 be real-valued finite sequences. We say that f_1 and f_2 are γ-equivalent if and only if

(Def. 6) (i) $\operatorname{len} f_1 = \operatorname{len} f_2$, and

(ii) $\operatorname{Mean} f_1 = \operatorname{Mean} f_2$.

One can check that the predicate is reflexive and symmetric.

Now we state the proposition:

(15) Let us consider real-valued finite sequences f_1, f_2. Suppose

(i) $\operatorname{dom} f_1 = \operatorname{dom} f_2$, and

(ii) $\sum f_1 = \sum f_2$.

Then f_1 and f_2 are γ-equivalent.

Let f be a real-valued finite sequence. The functors: $\operatorname{MeanLess} f$ and $\operatorname{MeanMore} f$ yielding subsets of $\mathbb{N}$ are defined by terms,

(Def. 7) $\{n$, where n is a natural number : $n \in \operatorname{dom} f$ and $f(n) < \operatorname{Mean} f\}$,

(Def. 8) $\{n$, where n is a natural number : $n \in \operatorname{dom} f$ and $f(n) > \operatorname{Mean} f\}$,

respectively.

Let us consider a real-valued finite sequence f. Now we state the propositions:

(16) $\operatorname{HetSet} f \subseteq \operatorname{dom} f$.

(17) $\operatorname{MeanLess} f \subseteq \operatorname{dom} f$.

(18) $\operatorname{MeanMore} f \subseteq \operatorname{dom} f$.

(19) $\operatorname{HetSet} f = \operatorname{MeanLess} f \cup \operatorname{MeanMore} f$.

Let f be a heterogeneous real-valued finite sequence. One can verify that $\operatorname{MeanLess} f$ is non empty and $\operatorname{MeanMore} f$ is non empty.

Let f be a homogeneous real-valued finite sequence.

Let us note that $\operatorname{MeanLess} f$ is empty and $\operatorname{MeanMore} f$ is empty.

Let us consider a heterogeneous non empty real-valued finite sequence f. Now we state the propositions:

(20) $\operatorname{MeanLess} f$ misses $\operatorname{MeanMore} f$.

(21) $\operatorname{Het} f \geqslant 2$. The theorem is a consequence of (19) and (20).

4. Auxiliary Replacement Function

Let f be a function, i, j be natural numbers, and a, b be objects. The functor $\operatorname{Replace}(f,i,j,a,b)$ yielding a function is defined by the term

(Def. 9) $(f +\cdot (i,a)) +\cdot (j,b)$.

Now we state the proposition:

(22) Let us consider a finite sequence f, natural numbers i, j, and objects a, b. Then $\operatorname{dom}\operatorname{Replace}(f,i,j,a,b) = \operatorname{dom} f$.

Let f be a real-valued finite sequence, i, j be natural numbers, and a, b be real numbers. Let us observe that $\operatorname{Replace}(f,i,j,a,b)$ is real-valued and finite sequence-like.

Now we state the propositions:

(23) Let us consider a real-valued finite sequence w, a real number r, and a natural number i. Suppose $i \in \operatorname{dom} w$. Then $w +\cdot (i,r) = ((w\restriction(i -' 1)) ^\frown \langle r\rangle) ^\frown w_{\downharpoonright i}$.

(24) Let us consider a real-valued finite sequence f, a natural number i, and a real number a. If $i \in \operatorname{dom} f$, then $\sum(f +\cdot (i,a)) = \sum f - f(i) + a$. The theorem is a consequence of (23).

(25) Let us consider a positive real-valued finite sequence f, a natural number i, and a real number a. Suppose $i \in \operatorname{dom} f$. Then $\prod(f +\cdot (i,a)) = \frac{\prod f \cdot a}{f(i)}$. The theorem is a consequence of (23).

(26) Let us consider a real-valued finite sequence f, natural numbers i, j, and real numbers a, b. Suppose

(i) i, $j \in \operatorname{dom} f$, and

(ii) $i \neq j$.

Then $\sum \operatorname{Replace}(f,i,j,a,b) = \sum f - f(i) - f(j) + a + b$. The theorem is a consequence of (24).

(27) Let us consider a positive real-valued finite sequence f, natural numbers i, j, and positive real numbers a, b. Suppose

(i) i, $j \in \operatorname{dom} f$, and

(ii) $i \neq j$.

Then $\prod \operatorname{Replace}(f,i,j,a,b) = \frac{\prod f \cdot a \cdot b}{f(i) \cdot f(j)}$. Proof: For every natural number n such that $n \in \operatorname{dom}(f +\cdot (i,a))$ holds $(f +\cdot (i,a))(n) > 0$ by [6, (30), (31), (32)]. $\prod \operatorname{Replace}(f,i,j,a,b) = \frac{\prod (f +\cdot (i,a)) \cdot b}{(f +\cdot (i,a))(j)}$. $\square$

(28) Let us consider a real-valued finite sequence f and natural numbers i, j. Suppose

(i) i, $j \in \operatorname{dom} f$, and

(ii) $i \neq j$.

Then f and $\mathrm{Replace}(f, i, j, \mathrm{Mean}\, f, (f(i)+f(j)-\mathrm{Mean}\, f))$ are γ-equivalent. The theorem is a consequence of (22) and (26).

(29) Let us consider a real-valued finite sequence f, natural numbers i, j, k, and real numbers a, b. Suppose

(i) i, j, $k \in \mathrm{dom}\, f$, and

(ii) $i \neq j$, and

(iii) $k \neq i$, and

(iv) $k \neq j$.

Then $(\mathrm{Replace}(f, i, j, a, b))(k) = f(k)$.

Let us consider a finite sequence f, natural numbers i, j, and objects a, b. Let us assume that i, $j \in \mathrm{dom}\, f$ and $i \neq j$. Now we state the propositions:

(30) $(\mathrm{Replace}(f, i, j, a, b))(j) = b$.

(31) $(\mathrm{Replace}(f, i, j, a, b))(i) = a$.

Now we state the propositions:

(32) Let us consider a real-valued finite sequence f and natural numbers i, j. Suppose

(i) i, $j \in \mathrm{dom}\, f$, and

(ii) $i \neq j$, and

(iii) $f(i) \neq \mathrm{Mean}\, f$, and

(iv) $f(j) \neq \mathrm{Mean}\, f$.

Then $\mathrm{Het}\, f > \mathrm{Het}\, \mathrm{Replace}(f, i, j, \mathrm{Mean}\, f, (f(i) + f(j) - \mathrm{Mean}\, f))$. The theorem is a consequence of (28), (31), (22), and (29).

(33) Let us consider positive non empty real-valued finite sequences f, g. Suppose

(i) $\mathrm{len}\, f = \mathrm{len}\, g$, and

(ii) $\prod f < \prod g$.

Then $\mathrm{GMean}\, f < \mathrm{GMean}\, g$.

(34) Let us consider a positive heterogeneous non empty real-valued finite sequence f. Then there exist natural numbers i, j such that

(i) i, $j \in \mathrm{dom}\, f$, and

(ii) $i \neq j$, and

(iii) $f(i) < \mathrm{Mean}\, f < f(j)$.

Let us consider a positive heterogeneous non empty real-valued finite sequence f and natural numbers i, j. Now we state the propositions:

(35) If $i, j \in \operatorname{dom} f$ and $i \neq j$ and $f(i) > \operatorname{Mean} f$, then $\operatorname{Replace}(f, i, j, \operatorname{Mean} f, (f(i) + f(j) - \operatorname{Mean} f))$ is positive. The theorem is a consequence of (22), (31), (30), and (29).

(36) If $i, j \in \operatorname{dom} f$ and $i \neq j$ and $f(j) > \operatorname{Mean} f$, then $\operatorname{Replace}(f, i, j, \operatorname{Mean} f, (f(i) + f(j) - \operatorname{Mean} f))$ is positive. The theorem is a consequence of (22), (31), (30), and (29).

Now we state the propositions:

(37) Let us consider a positive heterogeneous non empty real-valued finite sequence f. Then there exist natural numbers i, j such that

  (i) i, $j \in \operatorname{dom} f$, and

  (ii) $i \neq j$, and

  (iii) there exists a positive non empty real-valued finite sequence g such that $g = \operatorname{Replace}(f, i, j, \operatorname{Mean} f, (f(i)+f(j)-\operatorname{Mean} f))$ and $\operatorname{GMean} f < \operatorname{GMean} g$.

  The theorem is a consequence of (34), (22), (35), (27), and (33).

(38) Let us consider a heterogeneous non empty real-valued finite sequence f and natural numbers i, j. Suppose

  (i) $i =$ the element of $\operatorname{MeanLess} f$, and

  (ii) $j =$ the element of $\operatorname{MeanMore} f$.

  Then

  (iii) i, $j \in \operatorname{dom} f$, and

  (iv) $i \neq j$, and

  (v) $f(i) < \operatorname{Mean} f$, and

  (vi) $f(j) > \operatorname{Mean} f$.

(39) Let us consider a heterogeneous positive non empty real-valued finite sequence f and objects i, j. Suppose

  (i) $i \in \operatorname{MeanLess} f$, and

  (ii) $j \in \operatorname{MeanMore} f$.

  Then

  (iii) i, $j \in \operatorname{dom} f$, and

  (iv) $i \neq j$, and

  (v) $f(i) < \operatorname{Mean} f$, and

  (vi) $f(j) > \operatorname{Mean} f$.

Let us consider a positive heterogeneous non empty real-valued finite sequence f and natural numbers i, j. Now we state the propositions:

(40) Suppose $i, j \in \operatorname{dom} f$ and $i \neq j$ and $i \in \operatorname{MeanMore} f$ and $j \in \operatorname{MeanLess} f$. Then there exists a positive non empty real-valued finite sequence g such that

(i) $g = \operatorname{Replace}(f, i, j, \operatorname{Mean} f, (f(i) + f(j) - \operatorname{Mean} f))$, and

(ii) $\operatorname{GMean} f < \operatorname{GMean} g$.

The theorem is a consequence of (39), (22), (35), (27), and (33).

(41) Suppose $i, j \in \operatorname{dom} f$ and $i \neq j$ and $j \in \operatorname{MeanMore} f$ and $i \in \operatorname{MeanLess} f$. Then there exists a positive non empty real-valued finite sequence g such that

(i) $g = \operatorname{Replace}(f, i, j, \operatorname{Mean} f, (f(i) + f(j) - \operatorname{Mean} f))$, and

(ii) $\operatorname{GMean} f < \operatorname{GMean} g$.

The theorem is a consequence of (39), (22), (36), (27), and (33).

5. Homogenization of a Finite Sequence

Let f be a heterogeneous positive non empty real-valued finite sequence. The functor $\operatorname{Homogen} f$ yielding a real-valued finite sequence is defined by

(Def. 10) There exist natural numbers i, j such that

(i) $i =$ the element of $\operatorname{MeanLess} f$, and

(ii) $j =$ the element of $\operatorname{MeanMore} f$, and

(iii) $it = \operatorname{Replace}(f, i, j, \operatorname{Mean} f, (f(i) + f(j) - \operatorname{Mean} f))$.

Now we state the proposition:

(42) Let us consider a heterogeneous positive non empty real-valued finite sequence f. Then $\operatorname{dom} \operatorname{Homogen} f = \operatorname{dom} f$. The theorem is a consequence of (22).

Let f be a heterogeneous positive non empty real-valued finite sequence. Note that $\operatorname{Homogen} f$ is non empty.

Observe that $\operatorname{Homogen} f$ is positive.

Let us consider a heterogeneous positive non empty real-valued finite sequence f. Now we state the propositions:

(43) $\operatorname{Het} \operatorname{Homogen} f < \operatorname{Het} f$. The theorem is a consequence of (38) and (32).

(44) $\operatorname{Homogen} f$ and f are γ-equivalent. The theorem is a consequence of (38) and (28).

(45) $\operatorname{GMean} \operatorname{Homogen} f > \operatorname{GMean} f$. The theorem is a consequence of (39) and (41).

6. Cauchy Mean Theorem

Now we state the proposition:

(46) Let us consider a heterogeneous positive non empty real-valued finite sequence f. Then there exists a non empty homogeneous positive real-valued finite sequence g such that

(i) $\text{GMean}\, g > \text{GMean}\, f$, and

(ii) $\text{Mean}\, g = \text{Mean}\, f$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ there exists a positive non empty real-valued finite sequence g such that $\text{Het}\, g = \$_1$ and $\text{Mean}\, f = \text{Mean}\, g$ and $\text{GMean}\, g > \text{GMean}\, f$ and $\text{Het}\, g < \text{Het}\, f$. There exists a natural number k such that $\mathcal{P}[k]$. For every natural number k such that $k \neq 0$ and $\mathcal{P}[k]$ there exists a natural number n such that $n < k$ and $\mathcal{P}[n]$. $\mathcal{P}[0]$ from [3, Sch. 7]. □

Now we state the proposition:

(47) Inequality of arithmetic and geometric means:
Let us consider a non empty positive real-valued finite sequence f. Then $\text{GMean}\, f \leqslant \text{Mean}\, f$. The theorem is a consequence of (14), (13), and (46).

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Received June 13, 2014