

Differentiation in Normed Spaces¹

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Summary. In this article we formalized the Fréchet differentiation. It is defined as a generalization of the differentiation of a real-valued function of a single real variable to more general functions whose domain and range are subsets of normed spaces [14].

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The notation and terminology used in this paper have been introduced in the following articles: [5], [1], [4], [10], [6], [7], [16], [15], [11], [12], [13], [3], [8], [19], [20], [17], [18], [21], and [9].

Let us consider non empty sets D , E , F . Now we state the propositions:

- (1) There exists a function I from $(F^E)^D$ into $F^{D \times E}$ such that
 - (i) I is bijective, and
 - (ii) for every function f from D into F^E and for every elements d, e such that $d \in D$ and $e \in E$ holds $I(f)(d, e) = f(d)(e)$.
- (2) There exists a function I from $(F^E)^D$ into $F^{E \times D}$ such that
 - (i) I is bijective, and
 - (ii) for every function f from D into F^E and for every elements e, d such that $e \in E$ and $d \in D$ holds $I(f)(e, d) = f(d)(e)$.

Now we state the propositions:

- (3) Let us consider non-empty non empty finite sequences D , E and a non empty set F . Then there exists a function L from $(F^{\prod E})^{\prod D}$ into $F^{\prod(E \wedge D)}$ such that
 - (i) L is bijective, and

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- (ii) for every function f from $\prod D$ into $F\prod E$ and for every finite sequences e, d such that $e \in \prod E$ and $d \in \prod D$ holds $L(f)(e \wedge d) = f(d)(e)$.

The theorem is a consequence of (2). PROOF: Consider I being a function from $(F\prod E)\prod D$ into $F\prod E \times \prod D$ such that I is bijective and for every function f from $\prod D$ into $F\prod E$ and for every elements e, d such that $e \in \prod E$ and $d \in \prod D$ holds $I(f)(e, d) = f(d)(e)$. Consider J being a function from $\prod E \times \prod D$ into $\prod(E \wedge D)$ such that J is one-to-one and onto and for every finite sequences x, y such that $x \in \prod E$ and $y \in \prod D$ holds $J(x, y) = x \wedge y$. Reconsider $K = J^{-1}$ as a function from $\prod(E \wedge D)$ into $\prod E \times \prod D$. Define $\mathcal{G}(\text{element}) = I(\$_1) \cdot K$. For every element x such that $x \in (F\prod E)\prod D$ holds $\mathcal{G}(x) \in F\prod(E \wedge D)$ by [7, (5), (8), (128)]. Consider L being a function from $(F\prod E)\prod D$ into $F\prod(E \wedge D)$ such that for every element e such that $e \in (F\prod E)\prod D$ holds $L(e) = \mathcal{G}(e)$ from [7, Sch. 2]. For every function f from $\prod D$ into $F\prod E$ and for every finite sequences e, d such that $e \in \prod E$ and $d \in \prod D$ holds $L(f)(e \wedge d) = f(d)(e)$ by [9, (87)], [7, (26), (8), (5)]. \square

- (4) Let us consider non empty sets X, Y . Then there exists a function I from $X \times Y$ into $X \times \prod\langle Y \rangle$ such that

- (i) I is bijective, and
(ii) for every elements x, y such that $x \in X$ and $y \in Y$ holds $I(x, y) = \langle x, \langle y \rangle \rangle$.

PROOF: Consider J being a function from Y into $\prod\langle Y \rangle$ such that J is one-to-one and onto and for every element y such that $y \in Y$ holds $J(y) = \langle y \rangle$. Define $\mathcal{P}[\text{element}, \text{element}, \text{element}] \equiv \$_3 = \langle \$_1, \langle \$_2 \rangle \rangle$. For every elements x, y such that $x \in X$ and $y \in Y$ there exists an element z such that $z \in X \times \prod\langle Y \rangle$ and $\mathcal{P}[x, y, z]$ by [7, (5)], [9, (87)]. Consider I being a function from $X \times Y$ into $X \times \prod\langle Y \rangle$ such that for every elements x, y such that $x \in X$ and $y \in Y$ holds $\mathcal{P}[x, y, I(x, y)]$ from [5, Sch. 1]. \square

- (5) Let us consider a non-empty non empty finite sequence X and a non empty set Y . Then there exists a function K from $\prod X \times Y$ into $\prod(X \wedge \langle Y \rangle)$ such that

- (i) K is bijective, and
(ii) for every finite sequence x and for every element y such that $x \in \prod X$ and $y \in Y$ holds $K(x, y) = x \wedge \langle y \rangle$.

The theorem is a consequence of (4). PROOF: Consider I being a function from $\prod X \times Y$ into $\prod X \times \prod\langle Y \rangle$ such that I is bijective and for every element x and for every element y such that $x \in \prod X$ and $y \in Y$ holds $I(x, y) = \langle x, \langle y \rangle \rangle$. Consider J being a function from $\prod X \times \prod\langle Y \rangle$ into $\prod(X \wedge \langle Y \rangle)$ such that J is one-to-one and onto and for every finite sequences x, y such that $x \in \prod X$ and $y \in \prod\langle Y \rangle$ holds $J(x, y) = x \wedge y$. Set

$K = J \cdot I$. For every finite sequence x and for every element y such that $x \in \prod X$ and $y \in Y$ holds $K(x, y) = x \wedge \langle y \rangle$ by [9, (87)], [7, (5), (15)]. \square

(6) Let us consider a non empty set D , a non-empty non empty finite sequence E , and a non empty set F . Then there exists a function L from $(F \prod E)^D$ into $F \prod (E \wedge \langle D \rangle)$ such that

- (i) L is bijective, and
- (ii) for every function f from D into $F \prod E$ and for every finite sequence e and for every element d such that $e \in \prod E$ and $d \in D$ holds $L(f)(e \wedge \langle d \rangle) = f(d)(e)$.

The theorem is a consequence of (2) and (5). PROOF: Consider I being a function from $(F \prod E)^D$ into $F \prod E \times D$ such that I is bijective and for every function f from D into $F \prod E$ and for every elements e, d such that $e \in \prod E$ and $d \in D$ holds $I(f)(e, d) = f(d)(e)$. Consider J being a function from $\prod E \times D$ into $\prod (E \wedge \langle D \rangle)$ such that J is bijective and for every finite sequence x and for every element y such that $x \in \prod E$ and $y \in D$ holds $J(x, y) = x \wedge \langle y \rangle$. Reconsider $K = J^{-1}$ as a function from $\prod (E \wedge \langle D \rangle)$ into $\prod E \times D$. Define $\mathcal{G}(\text{element}) = I(\$1) \cdot K$. For every element x such that $x \in (F \prod E)^D$ holds $\mathcal{G}(x) \in F \prod (E \wedge \langle D \rangle)$ by [7, (5), (8), (128)]. Consider L being a function from $(F \prod E)^D$ into $F \prod (E \wedge \langle D \rangle)$ such that for every element e such that $e \in (F \prod E)^D$ holds $L(e) = \mathcal{G}(e)$ from [7, Sch. 2]. For every function f from D into $F \prod E$ and for every finite sequence e and for every element d such that $e \in \prod E$ and $d \in D$ holds $L(f)(e \wedge \langle d \rangle) = f(d)(e)$ by [7, (5), (26), (8)]. \square

In this paper S, T denote real normed spaces, f, f_1, f_2 denote partial functions from S to T , Z denotes a subset of S , and i, n denote natural numbers.

Let S be a set. Assume S is a real normed space. The functor $\text{NormSp}_{\mathbb{R}}(S)$ yielding a real normed space is defined by the term

(Def. 1) S .

Let S, T be real normed spaces. The functor $\text{diff}_{\text{SP}}(S, T)$ yielding a function is defined by

- (Def. 2) (i) $\text{dom } it = \mathbb{N}$, and
- (ii) $it(0) = T$, and
 - (iii) for every natural number i , $it(i+1) =$ the real norm space of bounded linear operators from S into $\text{NormSp}_{\mathbb{R}}(it(i))$.

Now we state the proposition:

- (7) (i) $(\text{diff}_{\text{SP}}(S, T))(0) = T$, and
- (ii) $(\text{diff}_{\text{SP}}(S, T))(1) =$ the real norm space of bounded linear operators from S into T , and

- (iii) $(\text{diff}_{\text{SP}}(S, T))(2) =$ the real norm space of bounded linear operators from S into the real norm space of bounded linear operators from S into T .

Let us consider a natural number i . Now we state the propositions:

- (8) $(\text{diff}_{\text{SP}}(S, T))(i)$ is a real normed space.
 (9) There exists a real normed space H such that
 (i) $H = (\text{diff}_{\text{SP}}(S, T))(i)$, and
 (ii) $(\text{diff}_{\text{SP}}(S, T))(i+1) =$ the real norm space of bounded linear operators from S into H .

Let S, T be real normed spaces and i be a natural number. The functor $\text{diff}_{\text{SP}}(S^i, T)$ yielding a real normed space is defined by the term

(Def. 3) $(\text{diff}_{\text{SP}}(S, T))(i)$.

Now we state the proposition:

- (10) Let us consider a natural number i . Then $\text{diff}_{\text{SP}}(S^{(i+1)}, T) =$ the real norm space of bounded linear operators from S into $\text{diff}_{\text{SP}}(S^i, T)$. The theorem is a consequence of (9).

Let S, T be real normed spaces and f be a set. Assume f is a partial function from S to T . The functor $\text{PartFuncs}(f, S, T)$ yielding a partial function from S to T is defined by the term

(Def. 4) f .

Let f be a partial function from S to T and Z be a subset of S . The functor $f'(Z)$ yielding a function is defined by

- (Def. 5) (i) $\text{dom } it = \mathbb{N}$, and
 (ii) $it(0) = f \upharpoonright Z$, and
 (iii) for every natural number i , $it(i+1) = (\text{PartFuncs}(it(i), S, \text{diff}_{\text{SP}}(S^i, T)))' \upharpoonright Z$.

Now we state the propositions:

- (11) (i) $f'(Z)(0) = f \upharpoonright Z$, and
 (ii) $f'(Z)(1) = (f \upharpoonright Z)' \upharpoonright Z$, and
 (iii) $f'(Z)(2) = ((f \upharpoonright Z)' \upharpoonright Z)' \upharpoonright Z$.

The theorem is a consequence of (7).

- (12) Let us consider a natural number i . Then $f'(Z)(i)$ is a partial function from S to $\text{diff}_{\text{SP}}(S^i, T)$. The theorem is a consequence of (7). **PROOF:** Define $\mathcal{P}[\text{natural number}] \equiv f'(Z)(\$_1)$ is a partial function from S to $\text{diff}_{\text{SP}}(S^{\mathbb{S}^1}, T)$. For every natural number n , $\mathcal{P}[n]$ from [2, Sch. 2]. \square

Let S, T be real normed spaces, f be a partial function from S to T , Z be a subset of S , and i be a natural number. The functor $\text{diff}_Z(f, i)$ yielding a partial function from S to $\text{diff}_{\text{SP}}(S^i, T)$ is defined by the term

(Def. 6) $f'(Z)(i)$.

Now we state the proposition:

(13) $\text{diff}_Z(f, i+1) = \text{diff}_Z(f, i)'|_Z$. The theorem is a consequence of (12) and (8).

Let S, T be real normed spaces, f be a partial function from S to T , Z be a subset of S , and n be a natural number. We say that f is differentiable n times on Z if and only if

(Def. 7) (i) $Z \subseteq \text{dom } f$, and

(ii) for every natural number i such that $i \leq n-1$ holds

$\text{PartFuncs}(f'(Z)(i), S, \text{diff}_{\text{SP}}(S^i, T))$ is differentiable on Z .

Now we state the propositions:

(14) f is differentiable n times on Z if and only if $Z \subseteq \text{dom } f$ and for every natural number i such that $i \leq n-1$ holds $\text{diff}_Z(f, i)$ is differentiable on Z .

(15) f is differentiable 1 times on Z if and only if $Z \subseteq \text{dom } f$ and $f|_Z$ is differentiable on Z . The theorem is a consequence of (14) and (7). PROOF: For every natural number i such that $i \leq 1-1$ holds $\text{diff}_Z(f, i)$ is differentiable on Z . \square

(16) f is differentiable 2 times on Z if and only if $Z \subseteq \text{dom } f$ and $f|_Z$ is differentiable on Z and $(f|_Z)'|_Z$ is differentiable on Z . The theorem is a consequence of (14), (7), and (11). PROOF: For every natural number i such that $i \leq 2-1$ holds $\text{diff}_Z(f, i)$ is differentiable on Z by [2, (14)]. \square

(17) Let us consider real normed spaces S, T , a partial function f from S to T , a subset Z of S , and a natural number n . Suppose f is differentiable n times on Z . Let us consider a natural number m . If $m \leq n$, then f is differentiable m times on Z .

(18) Let us consider a natural number n and a partial function f from S to T . If $1 \leq n$ and f is differentiable n times on Z , then Z is open. The theorem is a consequence of (17) and (15).

(19) Let us consider a natural number n and a partial function f from S to T . Suppose

(i) $1 \leq n$, and

(ii) f is differentiable n times on Z .

Let us consider a natural number i . Suppose $i \leq n$. Then

(iii) $(\text{diff}_{\text{SP}}(S, T))(i)$ is a real normed space, and

(iv) $f'(Z)(i)$ is a partial function from S to $\text{diff}_{\text{SP}}(S^i, T)$, and

(v) $\text{dom } \text{diff}_Z(f, i) = Z$.

The theorem is a consequence of (13) and (14).

(20) Let us consider a natural number n and partial functions f, g from S to T . Suppose

- (i) $1 \leq n$, and
- (ii) f is differentiable n times on Z , and
- (iii) g is differentiable n times on Z .

Let us consider a natural number i . Suppose $i \leq n$. Then $\text{diff}_Z(f + g, i) = \text{diff}_Z(f, i) + \text{diff}_Z(g, i)$. The theorem is a consequence of (18), (14), (19), (13), and (10). PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \leq n$, then $\text{diff}_Z(f + g, \$1) = \text{diff}_Z(f, \$1) + \text{diff}_Z(g, \$1)$. $\mathcal{P}[0]$ by [21, (27)]. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$ by [2, (11)], [11, (39)], [8, (5)]. For every natural number n , $\mathcal{P}[n]$ from [2, Sch. 2]. \square

(21) Let us consider a natural number n and partial functions f, g from S to T . Suppose

- (i) $1 \leq n$, and
- (ii) f is differentiable n times on Z , and
- (iii) g is differentiable n times on Z .

Then $f + g$ is differentiable n times on Z . The theorem is a consequence of (18), (14), (19), and (20). PROOF: For every natural number i such that $i \leq n - 1$ holds $\text{diff}_Z(f + g, i)$ is differentiable on Z by [11, (39)]. \square

(22) Let us consider a natural number n and partial functions f, g from S to T . Suppose

- (i) $1 \leq n$, and
- (ii) f is differentiable n times on Z , and
- (iii) g is differentiable n times on Z .

Let us consider a natural number i . Suppose $i \leq n$. Then $\text{diff}_Z(f - g, i) = \text{diff}_Z(f, i) - \text{diff}_Z(g, i)$. The theorem is a consequence of (18), (14), (19), (13), and (10). PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \leq n$, then $\text{diff}_Z(f - g, \$1) = \text{diff}_Z(f, \$1) - \text{diff}_Z(g, \$1)$. $\mathcal{P}[0]$ by [21, (30)]. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$ by [2, (11)], [11, (40)], [8, (5)]. For every natural number n , $\mathcal{P}[n]$ from [2, Sch. 2]. \square

(23) Let us consider a natural number n and partial functions f, g from S to T . Suppose

- (i) $1 \leq n$, and
- (ii) f is differentiable n times on Z , and
- (iii) g is differentiable n times on Z .

Then $f - g$ is differentiable n times on Z . The theorem is a consequence of (18), (14), (19), and (22). PROOF: For every natural number i such that $i \leq n - 1$ holds $\text{diff}_Z(f - g, i)$ is differentiable on Z by [11, (40)]. \square

(24) Let us consider a natural number n , a real number r , and a partial function f from S to T . Suppose

- (i) $1 \leq n$, and
- (ii) f is differentiable n times on Z .

Let us consider a natural number i . If $i \leq n$, then $\text{diff}_Z(r \cdot f, i) = r \cdot \text{diff}_Z(f, i)$. The theorem is a consequence of (18), (14), (19), (10), and (13). PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \leq n$, then $\text{diff}_Z(r \cdot f, \$1) = r \cdot \text{diff}_Z(f, \$1)$. $\mathcal{P}[0]$ by [21, (31)]. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$ by [2, (11)], [11, (41)]. For every natural number n , $\mathcal{P}[n]$ from [2, Sch. 2]. \square

(25) Let us consider a natural number n , a real number r , and a partial function f from S to T . Suppose

- (i) $1 \leq n$, and
- (ii) f is differentiable n times on Z .

Then $r \cdot f$ is differentiable n times on Z . The theorem is a consequence of (18), (14), (24), and (19). PROOF: For every natural number i such that $i \leq n - 1$ holds $\text{diff}_Z(r \cdot f, i)$ is differentiable on Z by [11, (41)]. \square

(26) Let us consider a natural number n and a partial function f from S to T . Suppose

- (i) $1 \leq n$, and
- (ii) f is differentiable n times on Z .

Let us consider a natural number i . Suppose $i \leq n$. Then $\text{diff}_Z(-f, i) = -\text{diff}_Z(f, i)$. The theorem is a consequence of (24).

(27) Let us consider a natural number n and a partial function f from S to T . Suppose

- (i) $1 \leq n$, and
- (ii) f is differentiable n times on Z .

Then $-f$ is differentiable n times on Z . The theorem is a consequence of (25).

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