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A Test for the Stability of Networks

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Summary. A complex polynomial is called a Hurwitz polynomial, if all its roots have a real part smaller than zero. This kind of polynomial plays an all-dominant role in stability checks of electrical (analog or digital) networks. In this article we prove that a polynomial p can be shown to be Hurwitz by checking whether the rational function $e(p)/o(p)$ can be realized as a reactance of one port, that is as an electrical impedance or admittance consisting of inductors and capacitors. Here $e(p)$ and $o(p)$ denote the even and the odd part of p [25].

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The notation and terminology used in this paper have been introduced in the following articles: [16], [14], [2], [3], [10], [4], [5], [22], [19], [21], [15], [1], [6], [17], [11], [12], [13], [18], [8], [26], [23], [20], [24], [9], [27], and [7].

1. Preliminaries

Now we state the propositions:

(1) Let us consider complex numbers x, y. If $\Im(x) = 0$ and $\Re(y) = 0$, then $\Re(\frac{x}{y}) = 0$.

(2) Let us consider a complex number a. Then $a \cdot \overline{a} = |a|^{\mathbf{2}}$.

One can check that there exists a polynomial of $\mathbb{C}_{\mathrm{F}}$ which is Hurwitz and 0 is even.

Now we state the propositions:

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(3) Let us consider an add-associative right zeroed right complementable associative distributive non empty double loop structure L, an even element k of $\mathbb{N}$, and an element x of L. Then $\mathrm{power}_L(-x, k) = \mathrm{power}_L(x, k)$.

(4) Let us consider an add-associative right zeroed right complementable associative distributive non empty double loop structure L, an odd element k of $\mathbb{N}$, and an element x of L. Then $\mathrm{power}_L(-x, k) = -\mathrm{power}_L(x, k)$. The theorem is a consequence of (3).

(5) Let us consider an even element k of $\mathbb{N}$ and an element x of $\mathbb{C}_{\mathrm{F}}$. If $\Re(x) = 0$, then $\Im(\mathrm{power}_{\mathbb{C}_{\mathrm{F}}}(x, k)) = 0$.

(6) Let us consider an odd element k of $\mathbb{N}$ and an element x of $\mathbb{C}_{\mathrm{F}}$. If $\Re(x) = 0$, then $\Re(\mathrm{power}_{\mathbb{C}_{\mathrm{F}}}(x, k)) = 0$.

2. Even and Odd Part of Polynomials

Let L be a non empty zero structure and p be a sequence of L. The functors the even part of p and the odd part of p yielding sequences of L are defined by the conditions, respectively.

(Def. 1) Let us consider an even natural number i. Then

- (i) (the even part of p)$(i) = p(i)$, and
- (ii) for every odd natural number i, (the even part of p)$(i) = 0_L$.

(Def. 2) Let us consider an even natural number i. Then

- (i) (the odd part of p)$(i) = 0_L$, and
- (ii) for every odd natural number i, (the odd part of p)$(i) = p(i)$.

Let p be a polynomial of L. Observe that the even part of p is finite-Support and the odd part of p is finite-Support. Now we state the propositions:

(7) Let us consider a non empty zero structure L. Then

- (i) the even part of $\mathbf{0}.\,L = \mathbf{0}.\,L$, and
- (ii) the odd part of $\mathbf{0}.\,L = \mathbf{0}.\,L$.

(8) Let us consider a non empty multiplicative loop with zero structure L. Then

- (i) the even part of $\mathbf{1}.\,L = \mathbf{1}.\,L$, and
- (ii) the odd part of $\mathbf{1}.\,L = \mathbf{0}.\,L$.

Let us consider a left zeroed right zeroed non empty additive loop structure L and a polynomial p of L. Now we state the propositions:

(9) (The even part of p) + (the odd part of p) $= p$.

(10) (The odd part of p) + (the even part of p) $= p$.

Let us consider an add-associative right zeroed right complementable non empty additive loop structure L and a polynomial p of L. Now we state the propositions:

(11) $p -$ the odd part of $p =$ the even part of p.

(12) $p -$ the even part of $p =$ the odd part of p.

Let us consider an add-associative right zeroed right complementable Abelian non empty additive loop structure L and a polynomial p of L. Now we state the propositions:

(13) (The even part of $p) - p = -$the odd part of p.

(14) (The odd part of $p) - p = -$the even part of p.

Let us consider an add-associative right zeroed right complementable Abelian non empty additive loop structure L and polynomials p, q of L. Now we state the propositions:

(15) The even part of $p + q =$ (the even part of p) + (the even part of q).

(16) The odd part of $p + q =$ (the odd part of p) + (the odd part of q).

Let us consider a well unital non empty double loop structure L and a polynomial p of L. Now we state the propositions:

(17) Suppose $\deg p$ is even. Then the even part of Leading-Monomial $p =$ Leading-Monomial p.

(18) If $\deg p$ is odd, then the even part of Leading-Monomial $p = \mathbf{0}. L$.

(19) If $\deg p$ is even, then the odd part of Leading-Monomial $p = \mathbf{0}. L$.

(20) Suppose $\deg p$ is odd. Then the odd part of Leading-Monomial $p =$ Leading-Monomial p.

Now we state the proposition:

(21) Let us consider a well unital add-associative right zeroed right complementable Abelian associative distributive non degenerated double loop structure L and a non zero polynomial p of L. Then $\deg$ the even part of $p \neq \deg$ the odd part of p. The theorem is a consequence of (9).

Let us consider a well unital add-associative right zeroed right complementable associative Abelian distributive non degenerated double loop structure L and a polynomial p of L. Now we state the propositions:

(22) (i) $\deg$ the even part of $p \leqslant \deg p$, and

(ii) $\deg$ the odd part of $p \leqslant \deg p$.

(23) $\deg p = \max(\deg$ the even part of $p, \deg$ the odd part of $p)$.

3. Even and Odd Polynomials and Rational Functions

Let L be a non empty additive loop structure and f be a function from L into L. We say that f is even if and only if

(Def. 3) Let us consider an element x of L. Then $f(-x) = f(x)$.

We say that f is odd if and only if

(Def. 4) Let us consider an element x of L. Then $f(-x) = -f(x)$.

Let L be a well unital non empty double loop structure and p be a polynomial of L. We say that p is even if and only if

(Def. 5) Polynomial-Function(L, p) is even.

We say that p is odd if and only if

(Def. 6) Polynomial-Function(L, p) is odd.

Let Z be a rational function of L. We say that Z is odd if and only if

(Def. 7) (i) $Z_{\mathbf{1}}$ is even and $Z_{\mathbf{2}}$ is odd, or

(ii) $Z_{\mathbf{1}}$ is odd and $Z_{\mathbf{2}}$ is even.

We introduce Z is even as an antonym for Z is odd.

Observe that there exists a polynomial of L which is even.

Let L be an add-associative right zeroed right complementable well unital non empty double loop structure. Let us note that there exists a polynomial of L which is odd.

Let L be a well unital add-associative right zeroed right complementable associative non degenerated double loop structure. Observe that there exists a polynomial of L which is non zero and even.

Let L be an add-associative right zeroed right complementable Abelian well unital non degenerated double loop structure. One can verify that there exists a polynomial of L which is non zero and odd.

Now we state the propositions:

(24) Let us consider a well unital non empty double loop structure L, an even polynomial p of L, and an element x of L. Then $\mathrm{eval}(p, -x) = \mathrm{eval}(p, x)$.

(25) Let us consider an add-associative right zeroed right complementable Abelian well unital non degenerated double loop structure L, an odd polynomial p of L, and an element x of L. Then $\mathrm{eval}(p, -x) = -\mathrm{eval}(p, x)$.

Let L be a well unital non empty double loop structure. One can verify that $\mathbf{0}.\,L$ is even.

Let L be an add-associative right zeroed right complementable well unital non empty double loop structure. One can verify that $\mathbf{0}.\,L$ is odd.

Let L be a well unital add-associative right zeroed right complementable associative non degenerated double loop structure. Note that $\mathbf{1}.\,L$ is even.

Let L be an Abelian add-associative right zeroed right complementable well unital left distributive non empty double loop structure and p, q be even polynomials of L. Let us note that $p + q$ is even.

Let p, q be odd polynomials of L. Let us note that $p + q$ is odd.

Let L be an Abelian add-associative right zeroed right complementable associative well unital distributive non degenerated double loop structure and p

be a polynomial of L. One can check that the even part of p is even and the odd part of p is odd.

Now we state the propositions:

(26) Let us consider an Abelian add-associative right zeroed right complementable well unital distributive non degenerated double loop structure L, an even polynomial p of L, an odd polynomial q of L, and an element x of L. If x is a common root of p and q, then $-x$ is a root of $p+q$. The theorem is a consequence of (24) and (25).

(27) Let us consider a Hurwitz polynomial p of $\mathbb{C}_{\mathrm{F}}$. Then the even part of p and the odd part of p have no common roots. The theorem is a consequence of (9) and (26).

4. Real Positive Polynomials and Rational Functions

Let p be a polynomial of $\mathbb{C}_{\mathrm{F}}$. We say that p is real if and only if

(Def. 8) Let us consider a natural number i. Then $p(i)$ is a real number.

We say that p is positive if and only if

(Def. 9) Let us consider an element x of $\mathbb{C}_{\mathrm{F}}$. If $\Re(x) > 0$, then $\Re(\mathrm{eval}(p, x)) > 0$.

Let us note that $\mathbf{0}.\mathbb{C}_{\mathrm{F}}$ is real and non positive and $\mathbf{1}.\mathbb{C}_{\mathrm{F}}$ is real and positive and there exists a polynomial of $\mathbb{C}_{\mathrm{F}}$ which is non zero, real, and positive and every polynomial of $\mathbb{C}_{\mathrm{F}}$ which is real is also real-valued.

Let p be a real polynomial of $\mathbb{C}_{\mathrm{F}}$. One can verify that the even part of p is real and the odd part of p is real.

Let L be a non empty additive loop structure and p be a polynomial of L. We say that p has all coefficients if and only if

(Def. 10) Let us consider a natural number i. If $i \leqslant \deg p$, then $p(i) \neq 0$.

Let p be a real polynomial of $\mathbb{C}_{\mathrm{F}}$. We say that p has positive coefficients if and only if

(Def. 11) Let us consider a natural number i. If $i \leqslant \deg p$, then $p(i) > 0$.

We say that p is negative coefficients if and only if

(Def. 12) Let us consider a natural number i. If $i \leqslant \deg p$, then $p(i) < 0$.

One can check that every real polynomial of $\mathbb{C}_{\mathrm{F}}$ which has positive coefficients has also all coefficients and every real polynomial of $\mathbb{C}_{\mathrm{F}}$ which is negative coefficients has also all coefficients and there exists a real polynomial of $\mathbb{C}_{\mathrm{F}}$ which is non constant and has positive coefficients.

Let p be a non zero real polynomial of $\mathbb{C}_{\mathrm{F}}$ with all coefficients. Let us note that the even part of p is non zero. Note that the odd part of p is non zero.

Let Z be a rational function of $\mathbb{C}_{\mathrm{F}}$. We say that Z is real if and only if

(Def. 13) Let us consider a natural number i. Then

(i) $Z_{\mathbf{1}}(i)$ is a real number, and

(ii) $Z_{\mathbf{2}}(i)$ is a real number.

We say that Z is positive if and only if

(Def. 14) Let us consider an element x of $\mathbb{C}_{\mathrm{F}}$. Suppose

(i) $\Re(x) > 0$, and

(ii) $\mathrm{eval}(Z_{\mathbf{2}}, x) \neq 0$.

Then $\Re(\mathrm{eval}(Z, x)) > 0$.

One can check that there exists a rational function of $\mathbb{C}_{\mathrm{F}}$ which is non zero, odd, real, and positive.

Let p_1 be a real polynomial of $\mathbb{C}_{\mathrm{F}}$ and p_2 be a non zero real polynomial of $\mathbb{C}_{\mathrm{F}}$. Let us note that $\langle p_1, p_2 \rangle$ is real as a rational function of $\mathbb{C}_{\mathrm{F}}$.

5. The Routh-Schur Stability Criterion

A one port function is a real positive rational function of $\mathbb{C}_{\mathrm{F}}$. A reactance one port function is an odd real positive rational function of $\mathbb{C}_{\mathrm{F}}$.

Let us consider a real polynomial p of $\mathbb{C}_{\mathrm{F}}$ and an element x of $\mathbb{C}_{\mathrm{F}}$. Now we state the propositions:

(28) If $\Re(x) = 0$, then $\Im(\mathrm{eval}(\text{the even part of } p, x)) = 0$.

(29) If $\Re(x) = 0$, then $\Re(\mathrm{eval}(\text{the odd part of } p, x)) = 0$.

Now we state the proposition:

(30) Let us consider a non constant real polynomial p of $\mathbb{C}_{\mathrm{F}}$ with positive coefficients. Suppose

(i) $\langle$the even part of p, the odd part of $p\rangle$ is positive, and

(ii) the even part of p and the odd part of p have no common roots.

Then

(iii) for every element x of $\mathbb{C}_{\mathrm{F}}$ such that $\Re(x) = 0$ and eval(the odd part of $p, x) \neq 0$ holds $\Re(\mathrm{eval}(\langle$the even part of p, the odd part of $p\rangle, x)) \geqslant 0$, and

(iv) (the even part of p) + (the odd part of p) is Hurwitz.

The theorem is a consequence of (28), (29), and (1).

Now we state the proposition:

(31) Routh-Schur stability criterion (for a single-input, single-output (SISO), linear time invariant (LTI) control system): Let us consider a non constant real polynomial p of $\mathbb{C}_{\mathrm{F}}$ with positive coefficients. Suppose

(i) $\langle$the even part of p, the odd part of $p\rangle$ is a one port function, and

(ii) degree(⟨the even part of p, the odd part of p⟩) = degree(p).

Then p is Hurwitz. The theorem is a consequence of (23), (30), and (9).

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