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The Derivations of Temporal Logic Formulas[1]

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Summary. This is a preliminary article to prove the completeness theorem of an extension of basic propositional temporal logic. We base it on the proof of completeness for basic propositional temporal logic given in [12]. We introduce n-ary connectives and prove their properties. We derive temporal logic formulas.

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The papers [14], [3], [1], [16], [6], [17], [8], [2], [7], [13], [4], [5], [11], [10], [15], and [9] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: A, B, p, q, r, s are elements of the LTLB-WFF, i, k, n are elements of $\mathbb{N}$, X is a subset of the LTLB-WFF, f, f_1 are finite sequences of elements of the LTLB-WFF, and g is a function from the LTLB-WFF into *Boolean*.

Let f be a finite sequence and let x be an empty set. One can check that $f(x)$ is empty.

We now state three propositions:

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(1) For every finite sequence f such that $\operatorname{len} f > 0$ and $n > 0$ holds $\operatorname{len}(f \restriction n) > 0$.

(2) For every finite sequence f such that $\operatorname{len} f = 0$ holds $f_{\restriction n} = f$.

(3) For all finite sequences f, g such that $\operatorname{rng} f = \operatorname{rng} g$ holds $\operatorname{len} f = 0$ iff $\operatorname{len} g = 0$.

Let us consider A, B. The functor $\mathrm{UN}(A, B)$ yields an element of the LTLB-WFF and is defined by:

(Def. 1) $\mathrm{UN}(A, B) = B \vee (A \,\&\&(A \,\mathcal{U}\, B))$.

One can prove the following proposition

(4) $\mathrm{VAL}_g(\top_t) = 1$.

Next we state the proposition

(5) $\mathrm{VAL}_g(p \vee q) = \mathrm{VAL}_g(p) \vee \mathrm{VAL}_g(q)$.

2. n-Argument Connectives and their Properties

Let us consider f. The functor conjunction f yielding a finite sequence of elements of the LTLB-WFF is defined as follows:

(Def. 2)(i) len conjunction $f = \operatorname{len} f$ and (conjunction f)(1) $= f(1)$ and for every i such that $1 \leq i < \operatorname{len} f$ holds (conjunction f)$(i+1) =$ (conjunction f)$_i \,\&\&\, f_{i+1}$ if $\operatorname{len} f > 0$,

(ii) conjunction $f = \langle \top_t \rangle$, otherwise.

Let us consider f, A. The functor implication(f, A) yielding a finite sequence of elements of the LTLB-WFF is defined as follows:

(Def. 3)(i) len implication$(f, A) = \operatorname{len} f$ and (implication(f, A))(1) $= \mathcal{G}(f_1) \Rightarrow A$ and for every i such that $1 \leq i < \operatorname{len} f$ holds (implication(f, A))$(i+1) = \mathcal{G}(f_{i+1}) \Rightarrow$ (implication(f, A))$_i$ if $\operatorname{len} f > 0$,

(ii) implication$(f, A) = \varepsilon_{(\text{the LTLB-WFF})}$, otherwise.

Let us consider f. The functor negation f yields a finite sequence of elements of the LTLB-WFF and is defined by:

(Def. 4) len negation $f = \operatorname{len} f$ and for every i such that $1 \leq i \leq \operatorname{len} f$ holds (negation f)$(i) = \neg(f_i)$.

Let us consider f. The functor next f yields a finite sequence of elements of the LTLB-WFF and is defined by:

(Def. 5) len next $f = \operatorname{len} f$ and for every i such that $1 \leq i \leq \operatorname{len} f$ holds (next f)$(i) = \mathcal{X}(f_i)$.

We now state a number of propositions:

(6) If $\operatorname{len} f > 0$, then (conjunction f)$_1 = f_1$.

(7) For every natural number i such that $1 \leq i < \operatorname{len} f$ holds (conjunction f)$_{i+1} =$ (conjunction f)$_i \,\&\&\, f_{i+1}$.

(8) For every natural number i such that $i \in \operatorname{dom} f$ holds $(\text{negation}\, f)_i = \neg(f_i)$.

(9) For every natural number i such that $i \in \operatorname{dom} f$ holds $(\text{next}\, f)_i = \mathcal{X}(f_i)$.

(10) $(\text{conjunction}(\varepsilon_{(\text{the LTLB-WFF})}))_{\text{len conjunction}(\varepsilon_{(\text{the LTLB-WFF})})} = \top_t$.

(11) $(\text{conjunction}\langle A\rangle)_{\text{len conjunction}\langle A\rangle} = A$.

(12) For every k such that $n \leq k$ holds $(\text{conjunction}\, f)(n) = (\text{conjunction}(f \restriction k))(n)$.

(13) For every k such that $n \leq k$ and $1 \leq n \leq \operatorname{len} f$ holds $(\text{conjunction}\, f)_n = (\text{conjunction}(f \restriction k))_n$.

(14) $\text{negation}\langle A\rangle = \langle \neg A\rangle$.

(15) $\text{negation}(f \frown \langle A\rangle) = (\text{negation}\, f) \frown \langle \neg A\rangle$.

(16) $\text{negation}(f \frown f_1) = (\text{negation}\, f) \frown \text{negation}\, f_1$.

(17) $\text{VAL}_g((\text{conjunction}(f \frown f_1))_{\text{len conjunction}(f \frown f_1)}) = \text{VAL}_g((\text{conjunction}\, f)_{\text{len conjunction}\, f}) \wedge \text{VAL}_g((\text{conjunction}\, f_1)_{\text{len conjunction}\, f_1})$.

(18) If $n \in \operatorname{dom} f$, then $\text{VAL}_g((\text{conjunction}\, f)_{\text{len conjunction}\, f}) = \text{VAL}_g((\text{conjunction}(f \restriction (n -' 1)))_{\text{len conjunction}(f \restriction (n -' 1))}) \wedge \text{VAL}_g(f_n) \wedge \text{VAL}_g((\text{conjunction}(f_{\downharpoonleft n}))_{\text{len conjunction}(f_{\downharpoonleft n})})$.

(19) $\text{VAL}_g((\text{conjunction}\, f)_{\text{len conjunction}\, f}) = 1$ iff for every natural number i such that $i \in \operatorname{dom} f$ holds $\text{VAL}_g(f_i) = 1$.

(20) $\text{VAL}_g(\neg((\text{conjunction negation}\, f)_{\text{len conjunction negation}\, f})) = 0$ iff for every natural number i such that $i \in \operatorname{dom} f$ holds $\text{VAL}_g(f_i) = 0$.

(21) If $\operatorname{rng} f = \operatorname{rng} f_1$, then $\text{VAL}_g((\text{conjunction}\, f)_{\text{len conjunction}\, f}) = \text{VAL}_g((\text{conjunction}\, f_1)_{\text{len conjunction}\, f_1})$.

3. Classical Tautologies of Temporal Language

Next we state a number of propositions:

(22) $p \Rightarrow \top_t$ is tautologically valid.

(23) $\neg\top_t \Rightarrow p$ is tautologically valid.

(24) $p \Rightarrow p$ is tautologically valid.

(25) $\neg\neg p \Rightarrow p$ is tautologically valid.

(26) $p \Rightarrow \neg\neg p$ is tautologically valid.

(27) $p \,\&\&\, q \Rightarrow p$ is tautologically valid.

(28) $p \,\&\&\, q \Rightarrow q$ is tautologically valid.

(29) For every natural number k such that $k \in \operatorname{dom} f$ holds $f_k \Rightarrow \neg((\text{conjunction negation}\, f)_{\text{len conjunction negation}\, f})$ is tautologically valid.

(30) If $\operatorname{rng} f \subseteq \operatorname{rng} f_1$, then $\neg((\text{conjunction negation}\, f)_{\text{len conjunction negation}\, f}) \Rightarrow \neg((\text{conjunction negation}\, f_1)_{\text{len conjunction negation}\, f_1})$ is tautologically valid.

(31) $\neg(p \Rightarrow q) \Rightarrow p$ is tautologically valid.

(32) $\neg(p \Rightarrow q) \Rightarrow \neg q$ is tautologically valid.

(33) $p \Rightarrow (q \Rightarrow p)$ is tautologically valid.

(34) $p \Rightarrow (q \Rightarrow (p \Rightarrow q))$ is tautologically valid.

(35) $\neg(p \,\&\&\, q) \Rightarrow \neg p \vee \neg q$ is tautologically valid.

(36) $\neg(p \vee q) \Rightarrow \neg p \,\&\&\, \neg q$ is tautologically valid.

(37) $\neg(p \,\&\&\, q) \Rightarrow (p \Rightarrow \neg q)$ is tautologically valid.

(38) $\neg(\top_t \,\&\&\, \neg A) \Rightarrow A$ is tautologically valid.

(39) $\neg(s \,\&\&\, q) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow \neg s))$ is tautologically valid.

(40) $(p \Rightarrow r) \Rightarrow ((p \Rightarrow s) \Rightarrow (p \Rightarrow r \,\&\&\, s))$ is tautologically valid.

(41) $\neg(p \,\&\&\, s) \Rightarrow \neg(r \,\&\&\, s \,\&\&(p \,\&\&\, q))$ is tautologically valid.

(42) $\neg(p \,\&\&\, s) \Rightarrow \neg(p \,\&\&\, q \,\&\&(r \,\&\&\, s))$ is tautologically valid.

(43) $(p \Rightarrow q \,\&\&\, \neg q) \Rightarrow \neg p$ is tautologically valid.

(44) $(q \Rightarrow p \,\&\&\, r) \Rightarrow ((p \Rightarrow s) \Rightarrow (q \Rightarrow s \,\&\&\, r))$ is tautologically valid.

(45) $(p \Rightarrow q) \Rightarrow ((r \Rightarrow s) \Rightarrow (p \,\&\&\, r \Rightarrow q \,\&\&\, s))$ is tautologically valid.

(46) $(p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow ((r \Rightarrow p) \Rightarrow (r \Rightarrow q)))$ is tautologically valid.

(47) $(p \Rightarrow q) \Rightarrow ((p \Rightarrow \neg r) \Rightarrow (p \Rightarrow \neg(q \Rightarrow r)))$ is tautologically valid.

(48) $(p \Rightarrow q \vee r) \Rightarrow ((r \Rightarrow s) \Rightarrow (p \Rightarrow q \vee s))$ is tautologically valid.

(49) $(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r))$ is tautologically valid.

(50) $(r \Rightarrow \mathrm{UN}(p, q)) \Rightarrow ((r \Rightarrow \neg p \,\&\&\, \neg q) \Rightarrow \neg r)$ is tautologically valid.

(51) $(r \Rightarrow \mathrm{UN}(p, q)) \Rightarrow ((r \Rightarrow \neg q \,\&\&\, \neg(p \,\mathcal{U}\, q)) \Rightarrow \neg r)$ is tautologically valid.

4. The Derivations of Temporal Logic Formulas within Classical Logic

One can prove the following propositions:

(52) If $X \vdash p \Rightarrow q$ and $X \vdash p \Rightarrow r$, then $X \vdash p \Rightarrow q \,\&\&\, r$.

(53) If $X \vdash p \Rightarrow q$ and $X \vdash r \Rightarrow s$, then $X \vdash p \,\&\&\, r \Rightarrow q \,\&\&\, s$.

(54) If $X \vdash p \Rightarrow q$ and $X \vdash p \Rightarrow r$ and $X \vdash r \Rightarrow p$, then $X \vdash r \Rightarrow q$.

(55) If $X \vdash p \Rightarrow q \,\&\&\, \neg q$, then $X \vdash \neg p$.

(56) If for every natural number i such that $i \in \mathrm{dom}\, f$ holds $\emptyset_{\text{the LTLB-WFF}} \vdash p \Rightarrow f_i$, then $\emptyset_{\text{the LTLB-WFF}} \vdash p \Rightarrow (\text{conjunction}\, f)_{\text{len conjunction}\, f}$.

(57) If for every natural number i such that $i \in \mathrm{dom}\, f$ holds $\emptyset_{\text{the LTLB-WFF}} \vdash f_i \Rightarrow p$, then $\emptyset_{\text{the LTLB-WFF}} \vdash \neg((\text{conjunction negation}\, f)_{\text{len conjunction negation}\, f}) \Rightarrow p$.

5. The Derivations of Temporal Logic Formulas

Next we state several propositions:

(58) $X \vdash (\mathcal{X}\, p \Rightarrow \mathcal{X}\, q) \Rightarrow \mathcal{X}(p \Rightarrow q)$.

(59) $X \vdash \mathcal{X}(p \,\&\&\, q) \Rightarrow \mathcal{X}\, p \,\&\&\, \mathcal{X}\, q$.

(60) $\emptyset_{\text{the LTLB-WFF}} \vdash (\text{conjunction next } f)_{\text{len conjunction next } f} \Rightarrow \mathcal{X}((\text{conjunction } f)_{\text{len conjunction } f})$.

(61) $X \vdash \mathcal{X}\, p \vee \mathcal{X}\, q \Rightarrow \mathcal{X}(p \vee q)$.

(62) $X \vdash \mathcal{X}(p \vee q) \Rightarrow \mathcal{X}\, p \vee \mathcal{X}\, q$.

(63) $X \vdash \neg(A\, \mathcal{U}\, B) \Rightarrow \mathcal{X} \neg \mathrm{UN}(A, B)$.

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