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The Rotation Group

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Summary. We introduce length-preserving linear transformations of Euclidean topological spaces. We also introduce rotation which preserves orientation (proper rotation) and reverses orientation (improper rotation). We show that every rotation that preserves orientation can be represented as a composition of base proper rotations. And finally, we show that every rotation that reverses orientation can be represented as a composition of proper rotations and one improper rotation.

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The papers [11], [35], [36], [8], [10], [9], [3], [7], [14], [2], [30], [4], [19], [12], [31], [24], [34], [13], [22], [17], [1], [20], [15], [16], [40], [38], [33], [25], [28], [37], [23], [6], [39], [18], [21], [32], [5], [26], [29], and [27] provide the terminology and notation for this paper.

1. Preliminaries

We adopt the following rules: x, X are sets, α, α_1, α_2, r, s are real numbers, and i, j, k, m, n are natural numbers.

We now state three propositions:

(1) Let K be a field, M be a square matrix over K of dimension n, and P be a permutation of $\mathrm{Seg}\, n$. Then $\mathrm{Det}(((M \cdot P)^{\mathrm{T}} \cdot P)^{\mathrm{T}}) = \mathrm{Det}\, M$ and for all i, j such that $\langle i, j\rangle \in$ the indices of M holds $((M \cdot P)^{\mathrm{T}} \cdot P)^{\mathrm{T}}_{i,j} = M_{P(i),P(j)}$.

(2) For every field K and for every diagonal square matrix M over K of dimension n holds $M^{\mathrm{T}} = M$.

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(3) For every real-valued finite sequence f and for every i such that $i \in \operatorname{dom} f$ holds $\sum^2(f +\cdot (i, r)) = (\sum^2 f - f(i)^{\mathbf{2}}) + r^{\mathbf{2}}$.

Let us consider X and let F be a function yielding function. We say that F is X-support-yielding if and only if:

(Def. 1) For every function f and for every x such that $f \in \operatorname{dom} F$ and $F(f)(x) \neq f(x)$ holds $x \in X$.

Let us consider X. One can check that there exists a function yielding function which is X-support-yielding.

Let us consider X and let Y be a subset of X. One can check that every function yielding function which is Y-support-yielding is also X-support-yielding.

Let X, Y be sets. Note that every function yielding function which is X-support-yielding and Y-support-yielding is also $X \cap Y$-support-yielding. Let f be an X-support-yielding function yielding function and let g be a Y-support-yielding function yielding function. Note that $f \cdot g$ is $X \cup Y$-support-yielding.

Let us consider n. Observe that there exists a function from $\mathcal{E}^n_{\rm T}$ into $\mathcal{E}^n_{\rm T}$ which is homogeneous.

Let us consider n, m. Observe that every function from $\mathcal{E}^n_{\rm T}$ into $\mathcal{E}^m_{\rm T}$ is finite sequence-yielding.

Let us consider n, m and let A be a matrix over $\mathbb{R}_{\rm F}$ of dimension $n \times m$. One can check that Mx2Tran A is additive.

Let us consider n and let A be a square matrix over $\mathbb{R}_{\rm F}$ of dimension n. Note that Mx2Tran A is homogeneous.

Let us consider n and let f, g be homogeneous functions from $\mathcal{E}^n_{\rm T}$ into $\mathcal{E}^n_{\rm T}$. Note that $f \cdot g$ is homogeneous.

2. Improper Rotation

In the sequel p, q are points of $\mathcal{E}^n_{\rm T}$.

Let us consider n, i. Let us assume that $i \in \operatorname{Seg} n$. The axial symmetry of i and n yields an invertible square matrix over $\mathbb{R}_{\rm F}$ of dimension n and is defined by the conditions (Def. 2).

(Def. 2)(i) (The axial symmetry of i and $n)_{i,i} = -1_{\mathbb{R}_{\rm F}}$, and

(ii) for all k, m such that $\langle k, m\rangle \in$ the indices of the axial symmetry of i and n holds if $k = m$ and $k \neq i$, then (the axial symmetry of i and $n)_{k,k} = 1_{\mathbb{R}_{\rm F}}$ and if $k \neq m$, then (the axial symmetry of i and $n)_{k,m} = 0_{\mathbb{R}_{\rm F}}$.

The following propositions are true:

(4) If $i \in \operatorname{Seg} n$, then Det (the axial symmetry of i and $n) = -1_{\mathbb{R}_{\rm F}}$.

(5) If i, $j \in \operatorname{Seg} n$ and $i \neq j$, then $({}^{@}p) \cdot$ (the axial symmetry of i and $n)_{\square,j} = p(j)$.

(6) If $i \in \operatorname{Seg} n$, then $({}^{@}p) \cdot$ (the axial symmetry of i and $n)_{\square,i} = -p(i)$.

(7) Suppose $i \in \operatorname{Seg} n$. Then
(i) the axial symmetry of i and n is diagonal, and
(ii) (the axial symmetry of i and $n)^{\smile}$ = the axial symmetry of i and n.

(8) If $i \in \operatorname{Seg} n$ and $i \neq j$, then (Mx2Tran (the axial symmetry of i and n))$(p)(j) = p(j)$.

(9) If $i \in \operatorname{Seg} n$, then (Mx2Tran (the axial symmetry of i and n))$(p)(i) = -p(i)$.

(10) If $i \in \operatorname{Seg} n$, then (Mx2Tran (the axial symmetry of i and n))$(p) = p +\cdot (i, -p(i))$.

(11) If $i \in \operatorname{Seg} n$, then Mx2Tran (the axial symmetry of i and n) is $\{i\}$-support-yielding.

(12) For all elements a, b of $\mathbb{R}_{\rm F}$ such that $a = \cos r$ and $b = \sin r$ holds Det (the $0_{\mathbb{R}_{\rm F}}$-block diagonal of $\langle \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, I^{n \times n}_{\mathbb{R}_{\rm F}} \rangle) = 1_{\mathbb{R}_{\rm F}}$.

3. Proper Rotation

Let us consider n, α and let us consider i, j. Let us assume that $1 \leq i < j \leq n$. The functor Rotation(i, j, n, α) yielding an invertible square matrix over $\mathbb{R}_{\rm F}$ of dimension n is defined by the conditions (Def. 3).

(Def. 3)(i) $(\text{Rotation}(i, j, n, \alpha))_{i,i} = \cos \alpha$,
(ii) $(\text{Rotation}(i, j, n, \alpha))_{j,j} = \cos \alpha$,
(iii) $(\text{Rotation}(i, j, n, \alpha))_{i,j} = \sin \alpha$,
(iv) $(\text{Rotation}(i, j, n, \alpha))_{j,i} = -\sin \alpha$, and
(v) for all k, m such that $\langle k, m \rangle \in$ the indices of Rotation(i, j, n, α) holds if $k = m$ and $k \neq i$ and $k \neq j$, then $(\text{Rotation}(i, j, n, \alpha))_{k,k} = 1_{\mathbb{R}_{\rm F}}$ and if $k \neq m$ and $\{k, m\} \neq \{i, j\}$, then $(\text{Rotation}(i, j, n, \alpha))_{k,m} = 0_{\mathbb{R}_{\rm F}}$.

We now state a number of propositions:

(13) If $1 \leq i < j \leq n$, then Det Rotation$(i, j, n, \alpha) = 1_{\mathbb{R}_{\rm F}}$.

(14) If $1 \leq i < j \leq n$ and $k \in \operatorname{Seg} n$ and $k \neq i$ and $k \neq j$, then $({}^{@}p) \cdot (\text{Rotation}(i, j, n, \alpha))_{\square,k} = p(k)$.

(15) If $1 \leq i < j \leq n$, then $({}^{@}p) \cdot (\text{Rotation}(i, j, n, \alpha))_{\square,i} = p(i) \cdot \cos \alpha + p(j) \cdot -\sin \alpha$.

(16) If $1 \leq i < j \leq n$, then $({}^{@}p) \cdot (\text{Rotation}(i, j, n, \alpha))_{\square,j} = p(i) \cdot \sin \alpha + p(j) \cdot \cos \alpha$.

(17) If $1 \leq i < j \leq n$, then Rotation$(i, j, n, \alpha_1) \cdot$ Rotation$(i, j, n, \alpha_2) =$ Rotation$(i, j, n, \alpha_1 + \alpha_2)$.

(18) If $1 \leq i < j \leq n$, then Rotation$(i, j, n, 0) = I^{n \times n}_{\mathbb{R}_{\rm F}}$.

(19) If $1 \leq i < j \leq n$, then Rotation(i, j, n, α) is orthogonal and $(\text{Rotation}(i, j, n, \alpha))^{\smile} = \text{Rotation}(i, j, n, -\alpha)$.

(20) If $1 \leq i < j \leq n$ and $k \neq i$ and $k \neq j$, then $(\mathrm{Mx2Tran\,Rotation}(i,j,n,\alpha))(p)(k) = p(k)$.

(21) If $1 \leq i < j \leq n$, then $(\mathrm{Mx2Tran\,Rotation}(i,j,n,\alpha))(p)(i) = p(i) \cdot \cos\alpha + p(j) \cdot -\sin\alpha$.

(22) If $1 \leq i < j \leq n$, then $(\mathrm{Mx2Tran\,Rotation}(i,j,n,\alpha))(p)(j) = p(i) \cdot \sin\alpha + p(j) \cdot \cos\alpha$.

(23) If $1 \leq i < j \leq n$, then $(\mathrm{Mx2Tran\,Rotation}(i,j,n,\alpha))(p) = (p \restriction (i -' 1)) \frown \langle p(i) \cdot \cos\alpha + p(j) \cdot -\sin\alpha \rangle \frown (p_{\lfloor i} \restriction (j -' i -' 1)) \frown \langle p(i) \cdot \sin\alpha + p(j) \cdot \cos\alpha \rangle \frown (p_{\lfloor j})$.

(24) If $1 \leq i < j \leq n$ and $s^{\mathbf{2}} \leq p(i)^{\mathbf{2}} + p(j)^{\mathbf{2}}$, then there exists α such that $(\mathrm{Mx2Tran\,Rotation}(i,j,n,\alpha))(p)(i) = s$.

(25) If $1 \leq i < j \leq n$ and $s^{\mathbf{2}} \leq p(i)^{\mathbf{2}} + p(j)^{\mathbf{2}}$, then there exists α such that $(\mathrm{Mx2Tran\,Rotation}(i,j,n,\alpha))(p)(j) = s$.

(26) If $1 \leq i < j \leq n$, then $\mathrm{Mx2Tran\,Rotation}(i,j,n,\alpha)$ is $\{i,j\}$-support-yielding.

4. Length-Preserving Linear Transformations

Let us consider n and let f be a function from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$. We say that f is rotation if and only if:

(Def. 4) $|p| = |f(p)|$.

One can prove the following proposition

(27) If $i \in \mathrm{Seg}\,n$, then Mx2Tran (the axial symmetry of i and n) is rotation.

Let us consider n and let f be a function from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$. We say that f is base rotation if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exists a finite sequence F of elements of the semigroup of functions onto the carrier of $\mathcal{E}^n_{\mathrm{T}}$ such that $f = \prod F$ and for every k such that $k \in \mathrm{dom}\,F$ there exist i, j, r such that $1 \leq i < j \leq n$ and $F(k) = \mathrm{Mx2Tran\,Rotation}(i,j,n,r)$.

Let us consider n. One can check that $\mathrm{id}_{\mathcal{E}^n_{\mathrm{T}}}$ is base rotation.

Let us consider n. One can check that there exists a function from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$ which is base rotation.

Let us consider n and let f, g be base rotation functions from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$. One can check that $f \cdot g$ is base rotation.

Next we state the proposition

(28) If $1 \leq i < j \leq n$, then $\mathrm{Mx2Tran\,Rotation}(i,j,n,r)$ is base rotation.

Let us consider n. Observe that every function from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$ which is base rotation is also homogeneous, additive, rotation, and homeomorphism.

Let us consider n and let f be a base rotation function from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$. Note that f^{-1} is base rotation.

Let us consider n and let f, g be rotation functions from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$. One can check that $f \cdot g$ is rotation.

In the sequel f, f_1, f_2 are homogeneous additive functions from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$.

Let us consider n and let us consider f. The functor AutMt f yields a square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension n and is defined as follows:

(Def. 6) $f = \mathrm{Mx2Tran}\,\mathrm{AutMt}\,f$.

Next we state several propositions:

(29) $\mathrm{AutMt}(f_1 \cdot f_2) = \mathrm{AutMt}\,f_2 \cdot \mathrm{AutMt}\,f_1$.

(30) Suppose $k \in X$ and $k \in \mathrm{Seg}\,n$. Then there exists f such that

(i) f is X-support-yielding and base rotation,

(ii) if $\overline{\overline{X \cap \mathrm{Seg}\,n}} > 1$, then $f(p)(k) \geq 0$, and

(iii) for every i such that $i \in X \cap \mathrm{Seg}\,n$ and $i \neq k$ holds $f(p)(i) = 0$.

(31) For every subset A of $\mathcal{E}^n_{\mathrm{T}}$ such that $f \restriction A = \mathrm{id}_A$ holds $f \restriction \mathrm{Lin}(A) = \mathrm{id}_{\mathrm{Lin}(A)}$.

(32) Let A be a subset of $\mathcal{E}^n_{\mathrm{T}}$. Suppose f is rotation and $f \restriction A = \mathrm{id}_A$. Let given i. Suppose $i \in \mathrm{Seg}\,n$ and the base finite sequence of n and $i \in \mathrm{Lin}(A)$. Then $f(p)(i) = p(i)$.

(33) Let f be a rotation function from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$. Suppose f is X-support-yielding and for every i such that $i \in X \cap \mathrm{Seg}\,n$ holds $p(i) = 0$. Then $f(p) = p$.

(34) If $i \in \mathrm{Seg}\,n$ and $n \geq 2$, then there exists f such that f is base rotation and $f(p) = p +\cdot (i, -p(i))$.

(35) If f is $\{i\}$-support-yielding and rotation, then AutMt f = the axial symmetry of i and n or $\mathrm{AutMt}\,f = I^{n \times n}_{\mathbb{R}_{\mathrm{F}}}$.

(36) If f_1 is rotation, then there exists f_2 such that f_2 is base rotation and $f_2 \cdot f_1$ is $\{n\}$-support-yielding.

5. Rotation Matrix Classification

The following three propositions are true:

(37) If f is rotation, then $\mathrm{Det}\,\mathrm{AutMt}\,f = 1_{\mathbb{R}_{\mathrm{F}}}$ iff f is base rotation.

(38) If f is rotation, then $\mathrm{Det}\,\mathrm{AutMt}\,f = 1_{\mathbb{R}_{\mathrm{F}}}$ or $\mathrm{Det}\,\mathrm{AutMt}\,f = -1_{\mathbb{R}_{\mathrm{F}}}$.

(39) If f_1 is rotation and $\mathrm{Det}\,\mathrm{AutMt}\,f_1 = -1_{\mathbb{R}_{\mathrm{F}}}$ and $i \in \mathrm{Seg}\,n$ and AutMt f_2 = the axial symmetry of i and n, then $f_1 \cdot f_2$ is base rotation.

Let us consider n and let f be a rotation homogeneous additive function from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$. One can check that AutMt f is orthogonal.

Let us consider n. One can verify that every function from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathcal{E}^n_{\mathrm{T}}$ which is homogeneous, additive, and rotation is also homeomorphism.

6. The Rotation Mapping a Given Point to Another Point

One can prove the following propositions:

(40) Suppose $n = 1$ and $|p| = |q|$. Then there exists f such that f is rotation and $f(p) = q$ either $\mathrm{AutMt}\, f =$ the axial symmetry of n and n or $\mathrm{AutMt}\, f = I^{n \times n}_{\mathbb{R}_{\mathrm{F}}}$.

(41) If $n \neq 1$ and $|p| = |q|$, then there exists f such that f is base rotation and $f(p) = q$.

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