FORMALIZED MATHEMATICS
Vol. 19, No. 4, Pages 227–232, 2011

Borel-Cantelli Lemma[1]

Peter Jaeger
Ludwig Maximilians University of Munich
Germany

Summary. This article is about the Borel-Cantelli Lemma in probability theory. Necessary definitions and theorems are given in [10] and [7].

MML identifier: `BOR_CANT`, version: `7.11.07 4.160.1126`

The notation and terminology used here have been introduced in the following papers: [17], [3], [4], [8], [13], [1], [2], [5], [15], [14], [21], [9], [12], [11], [16], [6], [20], [19], and [18].

For simplicity, we adopt the following rules: O_1 is a non empty set, S_1 is a σ-field of subsets of O_1, P_1 is a probability on S_1, A is a sequence of subsets of S_1, and n is an element of $\mathbb{N}$.

Let D be a set, let x, y be extended real numbers, and let a, b be elements of D. Then $(x > y \rightarrow a, b)$ is an element of D.

We now state two propositions:

(1) For every element k of $\mathbb{N}$ and for every element x of $\mathbb{R}$ such that k is odd and $x > 0$ and $x \leq 1$ holds $(-x\,\mathrm{ExpSeq}_{\mathbb{R}})(k+1)+(-x\,\mathrm{ExpSeq}_{\mathbb{R}})(k+2) \geq 0$.

(2) For every element x of $\mathbb{R}$ holds $1 + x \leq (\text{the function exp})(x)$.

Let s be a sequence of real numbers. The functor ExpFuncWithElementOf s yielding a sequence of real numbers is defined as follows:

(Def. 1) For every natural number d holds $(\mathrm{ExpFuncWithElementOf}\, s)(d) = \sum -s(d)\,\mathrm{ExpSeq}_{\mathbb{R}}$.

Next we state two propositions:

(3) (The partial product of $\mathrm{ExpFuncWithElementOf}(P_1 \cdot A))(n) =$ (the function exp)$(-(\sum_{\alpha=0}^{\kappa}(P_1 \cdot A)(\alpha))_{\kappa \in \mathbb{N}}(n))$.

[1]The author wants to thank Prof. F. Merkl for his kind support during the course of this work.

ISSN 1426–2630(p), 1898-9934(e)

(4) (The partial product of $P_1 \cdot A^{\mathbf{c}})(n) \leq$ (the partial product of ExpFuncWithElementOf$(P_1 \cdot A))(n)$.

Let n_1, n_2 be elements of $\mathbb{N}$. The functor SeqOfIFGT1(n_1, n_2) yielding a sequence of $\mathbb{N}$ is defined by:

(Def. 2) For every element n of $\mathbb{N}$ holds (SeqOfIFGT1$(n_1, n_2))(n) = (n > n_1 \to n + n_2, n)$.

Let k be an element of $\mathbb{N}$. The SeqOfIFGT2 k yields a sequence of $\mathbb{N}$ and is defined by:

(Def. 3) For every element n of $\mathbb{N}$ holds (the SeqOfIFGT2 $k)(n) = n + k$.

Let k be an element of $\mathbb{N}$. The SeqOfIFGT3 k yields a sequence of $\mathbb{N}$ and is defined as follows:

(Def. 4) For every element n of $\mathbb{N}$ holds (the SeqOfIFGT3 $k)(n) = (n > k \to 0, 1)$.

Let n_1, n_2 be elements of $\mathbb{N}$. The functor SeqOfIFGT4(n_1, n_2) yielding a sequence of $\mathbb{N}$ is defined as follows:

(Def. 5) For every element n of $\mathbb{N}$ holds (SeqOfIFGT4$(n_1, n_2))(n) = (n > n_1 + 1 \to n + n_2, n)$.

Let n_1, n_2 be elements of $\mathbb{N}$. One can verify that SeqOfIFGT1(n_1, n_2) is one-to-one and SeqOfIFGT4(n_1, n_2) is one-to-one.

Let n be an element of $\mathbb{N}$. Observe that the SeqOfIFGT2 n is one-to-one.

Let X be a set, let s be an element of $\mathbb{N}$, and let A be a sequence of subsets of X. The functor ShiftSeq(A, s) yielding a sequence of subsets of X is defined by:

(Def. 6) ShiftSeq$(A, s) = A \uparrow s$.

Let O_1 be a non empty set, let S_1 be a σ-field of subsets of O_1, let s be an element of $\mathbb{N}$, and let A be a sequence of subsets of S_1. The functor @ShiftSeq(A, s) yields a sequence of subsets of S_1 and is defined by:

(Def. 7) @ShiftSeq$(A, s) =$ ShiftSeq(A, s).

Next we state the proposition

(5)(i) For all sequences A, B of subsets of S_1 such that $n > n_1$ and $B = A \cdot$ SeqOfIFGT1(n_1, n_2) holds (the partial product of $P_1 \cdot B)(n) =$ (the partial product of $P_1 \cdot A)(n_1) \cdot$(the partial product of $P_1 \cdot$@ShiftSeq$(A, n_1 + n_2 + 1))(n - n_1 - 1)$, and

(ii) for all sequences A, B, C of subsets of S_1 and for every sequence e of $\mathbb{N}$ such that $n > n_1$ and $C = A \cdot e$ and $B = C \cdot$SeqOfIFGT1(n_1, n_2) holds (the partial Intersection of $B)(n) =$ (the partial Intersection of $C)(n_1) \cap$ (the partial Intersection of @ShiftSeq$(C, n_1 + n_2 + 1))(n - n_1 - 1)$.

Let O_1 be a non empty set, let S_1 be a σ-field of subsets of O_1, let P_1 be a probability on S_1, and let A be a sequence of subsets of S_1. We say that A is all independent w.r.t. P_1 if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let B be a sequence of subsets of S_1. Given a sequence e of $\mathbb{N}$ such that e is one-to-one and for every element n of $\mathbb{N}$ holds $A(e(n)) = B(n)$. Let n be an element of $\mathbb{N}$. Then (the partial product of $P_1 \cdot B)(n) = P_1$((the partial Intersection of $B)(n)$).

The following propositions are true:

(6) Suppose $n > n_1$ and A is all independent w.r.t. P_1. Then P_1((the partial Intersection of $A^{\mathbf{c}})(n_1) \cap$(the partial Intersection of @ShiftSeq$(A, n_1+n_2+1))(n-n_1-1)) =$ (the partial product of $P_1 \cdot A^{\mathbf{c}})(n_1) \cdot$(the partial product of $P_1 \cdot$ @ShiftSeq$(A, n_1 + n_2 + 1))(n - n_1 - 1)$.

(7) (The partial Intersection of $A^{\mathbf{c}})(n) =$ (the partial Union of $A)(n)^{\mathrm{c}}$.

(8) P_1((the partial Intersection of $A^{\mathbf{c}})(n)) = 1 - P_1$((the partial Union of $A)(n)$).

Let X be a set and let A be a sequence of subsets of X. The UnionShiftSeq A yielding a sequence of subsets of X is defined as follows:

(Def. 9) For every element n of $\mathbb{N}$ holds (the UnionShiftSeq $A)(n) = \bigcup$ ShiftSeq(A, n).

Let O_1 be a non empty set, let S_1 be a σ-field of subsets of O_1, and let A be a sequence of subsets of S_1. The @UnionShiftSeq A yields a sequence of subsets of S_1 and is defined as follows:

(Def. 10) The @UnionShiftSeq $A =$ the UnionShiftSeq A.

Let O_1 be a non empty set, let S_1 be a σ-field of subsets of O_1, and let A be a sequence of subsets of S_1. The @lim sup A yielding an event of S_1 is defined as follows:

(Def. 11) The @lim sup $A = \bigcap$(the @UnionShiftSeq A).

Let X be a set and let A be a sequence of subsets of X. The IntersectShiftSeq A yields a sequence of subsets of X and is defined as follows:

(Def. 12) For every element n of $\mathbb{N}$ holds (the IntersectShiftSeq $A)(n) =$ Intersection ShiftSeq(A, n).

Let O_1 be a non empty set, let S_1 be a σ-field of subsets of O_1, and let A be a sequence of subsets of S_1. The @IntersectShiftSeq A yielding a sequence of subsets of S_1 is defined as follows:

(Def. 13) The @IntersectShiftSeq $A =$ the IntersectShiftSeq A.

Let O_1 be a non empty set, let S_1 be a σ-field of subsets of O_1, and let A be a sequence of subsets of S_1. The @lim inf A yielding an event of S_1 is defined by:

(Def. 14) The @lim inf $A = \bigcup$(the @IntersectShiftSeq A).

The following propositions are true:

(9) (The @IntersectShiftSeq $A^{\mathbf{c}})(n) =$ (the @UnionShiftSeq $A)(n)^{\mathrm{c}}$.

(10) Suppose A is all independent w.r.t. P_1. Then P_1((the partial Intersection of $A^{\mathbf{c}}$)(n)) = (the partial product of $P_1 \cdot A^{\mathbf{c}}$)(n).

(11) Let X be a set and A be a sequence of subsets of X. Then

(i) the superior setsequence $A =$ the UnionShiftSeq A, and

(ii) the inferior setsequence $A =$ the IntersectShiftSeq A.

(12)(i) The superior setsequence $A =$ the @UnionShiftSeq A, and

(ii) the inferior setsequence $A =$ the @IntersectShiftSeq A.

Let O_1 be a non empty set, let S_1 be a σ-field of subsets of O_1, let P_1 be a probability on S_1, and let A be a sequence of subsets of S_1. The functor SumShiftSeq(P_1, A) yields a sequence of real numbers and is defined by:

(Def. 15) For every element n of $\mathbb{N}$ holds $(\text{SumShiftSeq}(P_1, A))(n) = \sum(P_1 \cdot$ @ShiftSeq$(A, n))$.

We now state several propositions:

(13) If $(\sum_{\alpha=0}^{\kappa}(P_1 \cdot A)(\alpha))_{\kappa \in \mathbb{N}}$ is convergent, then P_1(the @lim sup A) $= 0$ and $\lim \text{SumShiftSeq}(P_1, A) = 0$ and SumShiftSeq(P_1, A) is convergent.

(14)(i) For every set X and for every sequence A of subsets of X and for every element n of $\mathbb{N}$ and for every set x holds there exists an element k of $\mathbb{N}$ such that $x \in (\text{ShiftSeq}(A, n))(k)$ iff there exists an element k of $\mathbb{N}$ such that $k \geq n$ and $x \in A(k)$,

(ii) for every set X and for every sequence A of subsets of X and for every set x holds $x \in$ Intersection (the UnionShiftSeq A) iff for every element m of $\mathbb{N}$ there exists an element n of $\mathbb{N}$ such that $n \geq m$ and $x \in A(n)$,

(iii) for every sequence A of subsets of S_1 and for every set x holds $x \in \bigcap$(the @UnionShiftSeq A) iff for every element m of $\mathbb{N}$ there exists an element n of $\mathbb{N}$ such that $n \geq m$ and $x \in A(n)$,

(iv) for every set X and for every sequence A of subsets of X and for every set x holds $x \in \bigcup$(the IntersectShiftSeq A) iff there exists an element n of $\mathbb{N}$ such that for every element k of $\mathbb{N}$ such that $k \geq n$ holds $x \in A(k)$,

(v) for every sequence A of subsets of S_1 and for every set x holds $x \in \bigcup$(the @IntersectShiftSeq A) iff there exists an element n of $\mathbb{N}$ such that for every element k of $\mathbb{N}$ such that $k \geq n$ holds $x \in A(k)$, and

(vi) for every sequence A of subsets of S_1 and for every element x of O_1 holds $x \in \bigcup$(the @IntersectShiftSeq $A^{\mathbf{c}}$) iff there exists an element n of $\mathbb{N}$ such that for every element k of $\mathbb{N}$ such that $k \geq n$ holds $x \notin A(k)$.

(15)(i) $\limsup A =$ the @lim sup A,

(ii) $\liminf A =$ the @lim inf A,

(iii) the @lim inf $A^{\mathbf{c}} =$ (the @lim sup A)$^{\mathrm{c}}$,

(iv) P_1(the @lim inf $A^{\mathbf{c}}$) $+ P_1$(the @lim sup A) $= 1$, and

(v) $P_1(\liminf(A^{\mathbf{c}})) + P_1(\limsup A) = 1$.

(16)(i) If $(\sum_{\alpha=0}^{\kappa}(P_1 \cdot A)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent, then $P_1(\limsup A) = 0$ and $P_1(\liminf(A^{\mathbf{c}})) = 1$, and

(ii) if A is all independent w.r.t. P_1 and $(\sum_{\alpha=0}^{\kappa}(P_1 \cdot A)(\alpha))_{\kappa\in\mathbb{N}}$ is divergent to $+\infty$, then $P_1(\liminf(A^{\mathbf{c}})) = 0$ and $P_1(\limsup A) = 1$.

(17) If $(\sum_{\alpha=0}^{\kappa}(P_1 \cdot A)(\alpha))_{\kappa\in\mathbb{N}}$ is not convergent and A is all independent w.r.t. P_1, then $P_1(\liminf(A^{\mathbf{c}})) = 0$ and $P_1(\limsup A) = 1$.

(18) If A is all independent w.r.t. P_1, then $P_1(\liminf(A^{\mathbf{c}})) = 0$ or $P_1(\liminf(A^{\mathbf{c}})) = 1$ but $P_1(\limsup A) = 0$ or $P_1(\limsup A) = 1$.

(19) $(\sum_{\alpha=0}^{\kappa}(P_1 \cdot @\mathrm{ShiftSeq}(A, n_1+1))(\alpha))_{\kappa\in\mathbb{N}}(n) \leq (\sum_{\alpha=0}^{\kappa}(P_1 \cdot A)(\alpha))_{\kappa\in\mathbb{N}}(n_1 + 1 + n) - (\sum_{\alpha=0}^{\kappa}(P_1 \cdot A)(\alpha))_{\kappa\in\mathbb{N}}(n_1)$.

(20) P_1((the @IntersectShiftSeq $A^{\mathbf{c}})(n)) = 1 - P_1$((the @UnionShiftSeq $A)(n))$.

(21)(i) If $A^{\mathbf{c}}$ is all independent w.r.t. P_1, then P_1((the partial Intersection of $A)(n))$ = (the partial product of $P_1 \cdot A)(n)$, and

(ii) if A is all independent w.r.t. P_1, then $1 - P_1$((the partial Union of $A)(n))$ = (the partial product of $P_1 \cdot A^{\mathbf{c}})(n)$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[6] Fuguo Ge and Xiquan Liang. On the partial product of series and related basic inequalities. *Formalized Mathematics*, 13(**3**):413–416, 2005.

[7] Hans-Otto Georgii. *Stochastik, Einführung in die Wahrscheinlichkeitstheorie und Statistik*. deGruyter, Berlin, 2 edition, 2004.

[8] Adam Grabowski. On the Kuratowski limit operators. *Formalized Mathematics*, 11(**4**):399–409, 2003.

[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[10] Achim Klenke. *Wahrscheinlichkeitstheorie*. Springer-Verlag, Berlin, Heidelberg, 2006.

[11] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[12] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[13] Jarosław Kotowicz. The limit of a real function at infinity. *Formalized Mathematics*, 2(**1**):17–28, 1991.

[14] Andrzej Nędzusiak. Probability. *Formalized Mathematics*, 1(**4**):745–749, 1990.

[15] Andrzej Nędzusiak. σ-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.

[16] Konrad Raczkowski and Andrzej Nędzusiak. Series. *Formalized Mathematics*, 2(**4**):449–452, 1991.

[17] Piotr Rudnicki and Andrzej Trybulec. Abian's fixed point theorem. *Formalized Mathematics*, 6(**3**):335–338, 1997.

[18] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[19] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

[20] Bo Zhang, Hiroshi Yamazaki, and Yatsuka Nakamura. Limit of sequence of subsets. *Formalized Mathematics*, 13(**2**):347–352, 2005.

[21] Bo Zhang, Hiroshi Yamazaki, and Yatsuka Nakamura. Set sequences and monotone class. *Formalized Mathematics*, 13(**4**):435–441, 2005.

Received January 31, 2011