FORMALIZED MATHEMATICS
Vol. 19, No. 3, Pages 127–130, 2011

Mazur-Ulam Theorem

Artur Korniłowicz
Institute of Informatics
University of Białystok
Sosnowa 64, 15-887 Białystok, Poland

Summary. The Mazur-Ulam theorem [15] has been formulated as two registrations: `cluster bijective isometric -> midpoints-preserving Function of E,F;` and `cluster isometric midpoints-preserving -> Affine Function of E,F;` A proof given by Jussi Väisälä [23] has been formalized.

MML identifier: `MAZURULM`, version: `7.11.07 4.160.1126`

The notation and terminology used in this paper have been introduced in the following papers: [19], [18], [4], [5], [20], [11], [10], [14], [17], [1], [6], [16], [24], [25], [21], [13], [12], [22], [2], [9], [8], [3], and [7].

For simplicity, we use the following convention: E, F, G are real normed spaces, f is a function from E into F, g is a function from F into G, a, b are points of E, and t is a real number.

Let us note that $\mathbb{I}$ is closed.

Next we state four propositions:

(1) DYADIC is a dense subset of $\mathbb{I}$.

(2) $\overline{\mathrm{DYADIC}} = [0, 1]$.

(3) $a + a = 2 \cdot a$.

(4) $(a + b) - b = a$.

Let A be an upper bounded real-membered set and let r be a non negative real number. Observe that $r \circ A$ is upper bounded.

Let A be an upper bounded real-membered set and let r be a non positive real number. Note that $r \circ A$ is lower bounded.

Let A be a lower bounded real-membered set and let r be a non negative real number. Observe that $r \circ A$ is lower bounded.

© 2011 University of Białystok
ISSN 1426–2630(p), 1898-9934(e)

Let A be a lower bounded non empty real-membered set and let r be a non positive real number. One can check that $r \circ A$ is upper bounded.

Next we state three propositions:

(5) For every sequence f of real numbers holds $f + (\mathbb{N} \longmapsto t) = t + f$.

(6) For every real number r holds $\lim(\mathbb{N} \longmapsto r) = r$.

(7) For every convergent sequence f of real numbers holds $\lim(t + f) = t + \lim f$.

Let f be a convergent sequence of real numbers and let us consider t. One can check that $t + f$ is convergent.

Next we state three propositions:

(8) For every sequence f of real numbers holds $f \cdot (\mathbb{N} \longmapsto a) = f \cdot a$.

(9) $\lim(\mathbb{N} \longmapsto a) = a$.

(10) For every convergent sequence f of real numbers holds $\lim(f \cdot a) = \lim f \cdot a$.

Let f be a convergent sequence of real numbers and let us consider E, a. Note that $f \cdot a$ is convergent.

Let E, F be non empty normed structures and let f be a function from E into F. We say that f is isometric if and only if:

(Def. 1) For all points a, b of E holds $\|f(a) - f(b)\| = \|a - b\|$.

Let E, F be non empty RLS structures and let f be a function from E into F. We say that f is affine if and only if:

(Def. 2) For all points a, b of E and for every real number t such that $0 \leq t \leq 1$ holds $f((1-t) \cdot a + t \cdot b) = (1-t) \cdot f(a) + t \cdot f(b)$.

We say that f preserves midpoints if and only if:

(Def. 3) For all points a, b of E holds $f(\frac{1}{2} \cdot (a+b)) = \frac{1}{2} \cdot (f(a) + f(b))$.

Let E be a non empty normed structure. Observe that id_E is isometric.

Let E be a non empty RLS structure. Note that id_E is affine and preserves midpoints.

Let E be a non empty normed structure. Observe that there exists a unary operation on E which is bijective, isometric, and affine and preserves midpoints.

Next we state the proposition

(11) If f is isometric and g is isometric, then $g \cdot f$ is isometric.

Let us consider E and let f, g be isometric unary operations on E. One can verify that $g \cdot f$ is isometric.

The following proposition is true

(12) If f is bijective and isometric, then f^{-1} is isometric.

Let us consider E and let f be a bijective isometric unary operation on E. One can check that f^{-1} is isometric.

We now state the proposition

(13) If f preserves midpoints and g preserves midpoints, then $g \cdot f$ preserves midpoints.

Let us consider E and let f, g be unary operations on E preserving midpoints. Note that $g \cdot f$ preserves midpoints.

The following proposition is true

(14) If f is bijective and preserves midpoints, then f^{-1} preserves midpoints.

Let us consider E and let f be a bijective unary operation on E preserving midpoints. Observe that f^{-1} preserves midpoints.

Next we state the proposition

(15) If f is affine and g is affine, then $g \cdot f$ is affine.

Let us consider E and let f, g be affine unary operations on E. Observe that $g \cdot f$ is affine.

One can prove the following proposition

(16) If f is bijective and affine, then f^{-1} is affine.

Let us consider E and let f be a bijective affine unary operation on E. Observe that f^{-1} is affine.

Let E be a non empty RLS structure and let a be a point of E. The functor a-reflection yields a unary operation on E and is defined as follows:

(Def. 4) For every point b of E holds a-reflection$(b) = 2 \cdot a - b$.

The following proposition is true

(17) a-reflection $\cdot$ a-reflection $= \mathrm{id}_E$.

Let us consider E, a. Note that a-reflection is bijective.

We now state several propositions:

(18) a-reflection$(a) = a$ and for every b such that a-reflection$(b) = b$ holds $a = b$.

(19) a-reflection$(b) - a = a - b$.

(20) $\|a\text{-reflection}(b) - a\| = \|b - a\|$.

(21) a-reflection$(b) - b = 2 \cdot (a - b)$.

(22) $\|a\text{-reflection}(b) - b\| = 2 \cdot \|b - a\|$.

(23) a-reflection^{-1} $= a$-reflection.

Let us consider E, a. Observe that a-reflection is isometric.

Next we state the proposition

(24) If f is isometric, then f is continuous on $\mathrm{dom}\, f$.

Let us consider E, F. Observe that every function from E into F which is bijective and isometric also preserves midpoints.

Let us consider E, F. One can check that every function from E into F which is isometric and preserves midpoints is also affine.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):163–171, 1991.
[3] Józef Białas and Yatsuka Nakamura. Dyadic numbers and T_4 topological spaces. *Formalized Mathematics*, 5(**3**):361–366, 1996.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[8] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[9] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.
[10] Hiroshi Imura, Morishige Kimura, and Yasunari Shidama. The differentiable functions on normed linear spaces. *Formalized Mathematics*, 12(**3**):321–327, 2004.
[11] Artur Korniłowicz. Collective operations on number-membered sets. *Formalized Mathematics*, 17(**2**):99–115, 2009, doi: 10.2478/v10037-009-0011-0.
[12] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[13] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[14] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.
[15] Stanisław Mazur and Stanisław Ulam. Sur les transformationes isométriques d'espaces vectoriels normés. *C. R. Acad. Sci. Paris*, (194):946–948, 1932.
[16] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[17] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[18] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[19] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[20] Andrzej Trybulec. On the sets inhabited by numbers. *Formalized Mathematics*, 11(**4**):341–347, 2003.
[21] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[22] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[23] Jussi Väisälä. A proof of the Mazur-Ulam theorem. http://www.helsinki.fi/~jvaisala/mazurulam.pdf.
[24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[25] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received December 21, 2010