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Sequent Calculus, Derivability, Provability. Gödel's Completeness Theorem[1]

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Summary. Fifth of a series of articles laying down the bases for classical first order model theory. This paper presents multiple themes: first it introduces sequents, rules and sets of rules for a first order language L as L-dependent types. Then defines derivability and provability according to a set of rules, and gives several technical lemmas binding all those concepts. Following that, it introduces a fixed set D of derivation rules, and proceeds to convert them to Mizar functorial cluster registrations to give the user a slick interface to apply them.

The remaining goals summon all the definitions and results introduced in this series of articles. First: D is shown to be correct and having the requisites to deliver a sensible definition of Henkin model (see [18]). Second: as a particular application of all the machinery built thus far, the satisfiability and Gödel completeness theorems are shown when restricting to countable languages. The techniques used to attain this are inspired from [18], then heavily modified with the twofold goal of embedding them into the more flexible framework of a variable ruleset here introduced, and of proving completeness of a set of rules more sparing than the one there used; in particular the simpler ruleset allowed to avoid the definition and tractation of free occurence of a literal, a fact which, along with shortening proofs, is remarkable in its own right. A preparatory account of some of the ideas used in the proofs given here can be found in [15].

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The notation and terminology used here have been introduced in the following papers: [1], [3], [23], [22], [4], [6], [17], [11], [12], [13], [14], [7], [8], [5], [19], [16], [24], [2], [21], [9], [26], [28], [27], [20], [25], and [10].

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1. Formalization of the Notion of Derivability and Provability. Henkin's Theorem for Arbitrary Languages

For simplicity, we adopt the following convention: k, m, n denote natural numbers, m_1 denotes an element of $\mathbb{N}$, U denotes a non empty set, A, B, X, Y, Z, x, y, z denote sets, S denotes a language, s denotes an element of S, f, g denote functions, p_1, p_2, p_3, p_4 denote w.f.f. strings of S, P_1, P_2, P_3 denote subsets of AllFormulasOf S, t, t_1, t_2 denote termal strings of S, a denotes an of-atomic-formula element of S, l, l_1, l_2 denote literal elements of S, p denotes a finite sequence, and m_2 denotes a non zero natural number.

Let S be a language. The functor S-sequents is defined as follows:

(Def. 1) S-sequents $= \{\langle p_5, c_1\rangle; p_5$ ranges over subsets of AllFormulasOf S, c_1 ranges over w.f.f. strings of S: p_5 is finite$\}$.

Let S be a language. Note that S-sequents is non empty.

Let us consider S. Observe that S-sequents is relation-like.

Let S be a language and let x be a set. We say that x is S-sequent-like if and only if:

(Def. 2) $x \in S$-sequents.

Let us consider S, X. We say that X is S-sequents-like if and only if:

(Def. 3) $X \subseteq S$-sequents.

Let us consider S. One can check that every subset of S-sequents is S-sequents-like and every element of S-sequents is S-sequent-like.

Let S be a language. One can verify that there exists an element of S-sequents which is S-sequent-like and there exists a subset of S-sequents which is S-sequents-like.

Let us consider S. One can check that there exists a set which is S-sequent-like and there exists a set which is S-sequents-like.

Let S be a language. A rule of S is an element of $(2^{S\text{-sequents}})^{2^{S\text{-sequents}}}$.

Let S be a language. A rule set of S is a subset of $(2^{S\text{-sequents}})^{2^{S\text{-sequents}}}$.

For simplicity, we adopt the following rules: D, D_1, D_2, D_3 denote rule sets of S, R denotes a rule of S, S_1, S_2, S_3 denote subsets of S-sequents, s_1, s_2, s_3 denote elements of S-sequents, S_4, S_5 denote S-sequents-like sets, and S_6, S_7 denote S-sequent-like sets.

Let us consider A, B and let X be a subset of B^A. One can check that $\bigcup X$ is relation-like.

Let S be a language and let D be a rule set of S. One can check that $\bigcup D$ is relation-like.

Let us consider S, D. The functor OneStep D yielding a rule of S is defined as follows:

(Def. 4) For every element S_8 of $2^{S\text{-sequents}}$ holds $(\text{OneStep}\,D)(S_8) = \bigcup((\bigcup D)^{\circ}\{S_8\})$.

Let us consider S, D, m. The functor (m, D)-derivables yields a rule of S and is defined by:

(Def. 5) (m, D)-derivables $= (\text{OneStep}\, D)^m$.

Let S be a language, let D be a rule set of S, and let S_9, S_{10} be sets. We say that S_{10} is (S_9, D)-derivable if and only if:

(Def. 6) $S_{10} \subseteq \bigcup(((\text{OneStep}\, D)^*)^\circ\{S_9\})$.

Let us consider m, S, D and let S_1, s_1 be sets. We say that s_1 is (m, S_1, D)-derivable if and only if:

(Def. 7) $s_1 \in (m, D)$-derivables(S_1).

Let us consider S, D. The functor D-iterators yielding a family of subsets of $2^{S\text{-sequents}} \times 2^{S\text{-sequents}}$ is defined as follows:

(Def. 8) D-iterators $= \{(\text{OneStep}\, D)^{m_1}\}$.

Let us consider S, R. We say that R is isotone if and only if:

(Def. 9) If $S_2 \subseteq S_3$, then $R(S_2) \subseteq R(S_3)$.

Let us consider S. Observe that there exists a rule of S which is isotone.

Let us consider S, D. We say that D is isotone if and only if:

(Def. 10) For all S_2, S_3, f such that $S_2 \subseteq S_3$ and $f \in D$ there exists g such that $g \in D$ and $f(S_2) \subseteq g(S_3)$.

Let us consider S and let M be an isotone rule of S. One can verify that $\{M\}$ is isotone.

Let us consider S. One can verify that there exists a rule set of S which is isotone.

In the sequel K, K_1 are isotone rule sets of S.

Let S be a language, let D be a rule set of S, and let S_1 be a set. We say that S_1 is D-derivable if and only if:

(Def. 11) S_1 is $(\emptyset, D)$-derivable.

Let us consider S, D. One can verify that every set which is D-derivable is also $(\emptyset, D)$-derivable and every set which is $(\emptyset, D)$-derivable is also D-derivable.

Let us consider S, D and let S_1 be an empty set. One can verify that every set which is (S_1, D)-derivable is also D-derivable.

Let us consider S, D, X and let p_2 be a set. We say that p_2 is (X, D)-provable if and only if:

(Def. 12) $\{\langle X, p_2\rangle\}$ is D-derivable or there exists a set s_1 such that $(s_1)_{\mathbf{1}} \subseteq X$ and $(s_1)_{\mathbf{2}} = p_2$ and $\{s_1\}$ is D-derivable.

Let us consider S, D, X, x. Let us observe that x is (X, D)-provable if and only if:

(Def. 13) There exists a set s_1 such that $(s_1)_{\mathbf{1}} \subseteq X$ and $(s_1)_{\mathbf{2}} = x$ and $\{s_1\}$ is D-derivable.

Let us consider S, D, R. We say that R is D-macro if and only if:

(Def. 14) For every subset S_8 of S-sequents holds $R(S_8)$ is (S_8, D)-derivable.

Let us consider S, D and let P_1 be a set. The functor (P_1, D)-termEq is defined as follows:

(Def. 15) (P_1, D)-termEq $= \{\langle t_1, t_2\rangle; t_1$ ranges over termal strings of S, t_2 ranges over termal strings of S: $\langle$TheEqSymbOf $S\rangle \frown t_1 \frown t_2$ is (P_1, D)-provable$\}$.

Let us consider S, D and let P_1 be a set. We say that P_1 is D-expanded if and only if:

(Def. 16) If x is (P_1, D)-provable, then $\{x\} \subseteq P_1$.

Let us consider S, x. We say that x is S-null if and only if:

(Def. 17) Not contradiction.

Let us consider S, D and let P_1 be a set. Then (P_1, D)-termEq is a binary relation on AllTermsOf S.

Let us consider S, p_2 and let P_2, P_3 be finite subsets of AllFormulasOf S. One can check that $\langle P_2 \cup P_3, p_2\rangle$ is S-sequent-like.

Let us consider S, let x be an empty set, and let p_2 be a w.f.f. string of S. Then $\langle x, p_2\rangle$ is an element of S-sequents.

Let us consider S. Note that $\emptyset \cap S$ is S-sequents-like.

Let us consider S. One can verify that there exists a set which is S-null.

Let us consider S. One can check that every set which is S-sequent-like is also S-null.

Let us consider S. One can check that every element of S-sequents is S-null.

Let us consider m, S, D, X. One can verify that (m, D)-derivables(X) is S-sequents-like.

Let us consider S, Y and let X be an S-sequents-like set. One can verify that $X \cap Y$ is S-sequents-like.

Let us consider S, D, m, X. Note that every set which is (m, X, D)-derivable is also S-sequent-like.

Let us consider S, D and let P_2, P_3 be sets. Observe that every set which is $(P_2 \setminus P_3, D)$-provable is also (P_2, D)-provable.

Let us consider S, D and let P_2, P_3 be sets. Observe that every set which is $(P_2 \setminus P_3, D)$-provable is also $(P_2 \cup P_3, D)$-provable.

Let us consider S, D and let P_2, P_3 be sets. Observe that every set which is $(P_2 \cap P_3, D)$-provable is also (P_2, D)-provable.

Let us consider S, D, let X be a set, and let x be a subset of X. Note that every set which is (x, D)-provable is also (X, D)-provable.

Let us consider S, let p_5 be a finite subset of AllFormulasOf S, and let p_2 be a w.f.f. string of S. One can check that $\langle p_5, p_2\rangle$ is S-sequent-like.

Let us consider S and let p_3, p_4 be w.f.f. strings of S. Observe that $\langle\{p_3\}, p_4\rangle$ is S-sequent-like. Let p_6 be a w.f.f. string of S. Note that $\langle\{p_3, p_4\}, p_6\rangle$ is S-sequent-like.

Let us consider S, p_3, p_4 and let P_1 be a finite subset of AllFormulasOf S. One can verify that $\langle P_1 \cup \{p_3\}, p_4\rangle$ is S-sequent-like.

Let us consider S, D. Note that there exists a subset of AllFormulasOf S which is D-expanded.

Let us consider S, D. Observe that there exists a set which is D-expanded.

Let S_1 be a set, let S be a language, and let s_1 be an S-null set. We say that s_1 rule 0 S_1 if and only if:

(Def. 18) $(s_1)_{\mathbf{2}} \in (s_1)_{\mathbf{1}}$.

We say that s_1 rule 1 S_1 if and only if:

(Def. 19) There exists a set y such that $y \in S_1$ and $y_{\mathbf{1}} \subseteq (s_1)_{\mathbf{1}}$ and $(s_1)_{\mathbf{2}} = y_{\mathbf{2}}$.

We say that s_1 rule 2 S_1 if and only if:

(Def. 20) $(s_1)_{\mathbf{1}}$ is empty and there exists a termal string t of S such that $(s_1)_{\mathbf{2}} = \langle \text{TheEqSymbOf}\, S\rangle \frown t \frown t$.

We say that s_1 rule 3a S_1 if and only if the condition (Def. 21) is satisfied.

(Def. 21) There exist termal strings t, t_1, t_2 of S and there exists a set x such that $x \in S_1$ and $(s_1)_{\mathbf{1}} = x_{\mathbf{1}} \cup \{\langle \text{TheEqSymbOf}\, S\rangle \frown t_1 \frown t_2\}$ and $x_{\mathbf{2}} = \langle \text{TheEqSymbOf}\, S\rangle \frown t \frown t_1$ and $(s_1)_{\mathbf{2}} = \langle \text{TheEqSymbOf}\, S\rangle \frown t \frown t_2$.

We say that s_1 rule 3b S_1 if and only if:

(Def. 22) There exist termal strings t_1, t_2 of S such that $(s_1)_{\mathbf{1}} = \{\langle \text{TheEqSymbOf}\, S\rangle \frown t_1 \frown t_2\}$ and $(s_1)_{\mathbf{2}} = \langle \text{TheEqSymbOf}\, S\rangle \frown t_2 \frown t_1$.

We say that s_1 rule 3d S_1 if and only if the condition (Def. 23) is satisfied.

(Def. 23) There exists a low-compounding element s of S and there exist $|\text{ar}\, s|$-element elements T, U of $(\text{AllTermsOf}\, S)^*$ such that
   (i) s is operational,
   (ii) $(s_1)_{\mathbf{1}} = \{\langle \text{TheEqSymbOf}\, S\rangle \frown T_1(j) \frown U_1(j); j$ ranges over elements of $\text{Seg}|\text{ar}\, s|, T_1$ ranges over functions from $\text{Seg}|\text{ar}\, s|$ into $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}, U_1$ ranges over functions from $\text{Seg}|\text{ar}\, s|$ into $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\} : T_1 = T \ \wedge \ U_1 = U\}$, and
   (iii) $(s_1)_{\mathbf{2}} = \langle \text{TheEqSymbOf}\, S\rangle \frown (s\text{-compound}\, T) \frown (s\text{-compound}\, U)$.

We say that s_1 rule 3e S_1 if and only if the condition (Def. 24) is satisfied.

(Def. 24) There exists a relational element s of S and there exist $|\text{ar}\, s|$-element elements T, U of $(\text{AllTermsOf}\, S)^*$ such that
   (i) $(s_1)_{\mathbf{1}} = \{s\text{-compound}\, T\} \cup \{\langle \text{TheEqSymbOf}\, S\rangle \frown T_1(j) \frown U_1(j); j$ ranges over elements of $\text{Seg}|\text{ar}\, s|, T_1$ ranges over functions from $\text{Seg}|\text{ar}\, s|$ into $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}, U_1$ ranges over functions from $\text{Seg}|\text{ar}\, s|$ into $(\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\} : T_1 = T \ \wedge \ U_1 = U\}$, and
   (ii) $(s_1)_{\mathbf{2}} = s\text{-compound}\, U$.

We say that s_1 rule 4 S_1 if and only if the condition (Def. 25) is satisfied.

(Def. 25) There exists a literal element l of S and there exists a w.f.f. string p_2 of S and there exists a termal string t of S such that $(s_1)_{\mathbf{1}} = \{(l,t)\,\mathrm{SubstIn}\,p_2\}$ and $(s_1)_{\mathbf{2}} = \langle l\rangle \frown p_2$.

We say that s_1 rule 5 S_1 if and only if:

(Def. 26) There exist literal elements v_1, v_2 of S and there exist x, p such that $(s_1)_{\mathbf{1}} = x \cup \{\langle v_1\rangle \frown p\}$ and v_2 is $x \cup \{p\} \cup \{s_{1\mathbf{2}}\}$-absent and $\langle x \cup \{(v_1\,\mathrm{SubstWith}\,v_2)(p)\},\, (s_1)_{\mathbf{2}}\rangle \in S_1$.

We say that s_1 rule 6 S_1 if and only if the condition (Def. 27) is satisfied.

(Def. 27) There exist sets y_1, y_2 and there exist w.f.f. strings p_3, p_4 of S such that $y_1, y_2 \in S_1$ and $(y_1)_{\mathbf{1}} = (y_2)_{\mathbf{1}} = (s_1)_{\mathbf{1}}$ and $(y_1)_{\mathbf{2}} = \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_3 \frown p_3$ and $(y_2)_{\mathbf{2}} = \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_4 \frown p_4$ and $(s_1)_{\mathbf{2}} = \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_3 \frown p_4$.

We say that s_1 rule 7 S_1 if and only if:

(Def. 28) There exists a set y and there exist w.f.f. strings p_3, p_4 of S such that $y \in S_1$ and $y_{\mathbf{1}} = (s_1)_{\mathbf{1}}$ and $y_{\mathbf{2}} = \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_3 \frown p_4$ and $(s_1)_{\mathbf{2}} = \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_4 \frown p_3$.

We say that s_1 rule 8 S_1 if and only if the condition (Def. 29) is satisfied.

(Def. 29) There exist sets y_1, y_2 and there exist w.f.f. strings p_2, p_3, p_4 of S such that $y_1, y_2 \in S_1$ and $(y_1)_{\mathbf{1}} = (y_2)_{\mathbf{1}}$ and $(y_1)_{\mathbf{2}} = p_3$ and $(y_2)_{\mathbf{2}} = \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_3 \frown p_4$ and $\{p_2\} \cup (s_1)_{\mathbf{1}} = (y_1)_{\mathbf{1}}$ and $(s_1)_{\mathbf{2}} = \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_2 \frown p_2$.

We say that s_1 rule 9 S_1 if and only if:

(Def. 30) There exists a set y and there exists a w.f.f. string p_2 of S such that $y \in S_1$ and $(s_1)_{\mathbf{2}} = p_2$ and $y_{\mathbf{1}} = (s_1)_{\mathbf{1}}$ and $y_{\mathbf{2}} = \mathrm{xnot}\,\mathrm{xnot}\,p_2$.

Let S be a language. The functor $\mathrm{P0}\,S$ yielding a relation between $2^{S\text{-sequents}}$ and S-sequents is defined by:

(Def. 31) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1\rangle \in \mathrm{P0}\,S$ iff s_1 rule 0 S_1.

The functor $\mathrm{P1}\,S$ yields a relation between $2^{S\text{-sequents}}$ and S-sequents and is defined as follows:

(Def. 32) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1\rangle \in \mathrm{P1}\,S$ iff s_1 rule 1 S_1.

The functor $\mathrm{P2}\,S$ yields a relation between $2^{S\text{-sequents}}$ and S-sequents and is defined as follows:

(Def. 33) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1\rangle \in \mathrm{P2}\,S$ iff s_1 rule 2 S_1.

The functor $\mathrm{P3a}\,S$ yielding a relation between $2^{S\text{-sequents}}$ and S-sequents is defined as follows:

(Def. 34) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1\rangle \in \mathrm{P3a}\,S$ iff s_1 rule 3a S_1.

The functor $\mathrm{P3b}\,S$ yields a relation between $2^{S\text{-sequents}}$ and S-sequents and is defined as follows:

(Def. 35) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1\rangle \in \mathrm{P3b}\,S$ iff s_1 rule 3b S_1.

The functor $\mathrm{P3d}\,S$ yields a relation between $2^{S\text{-sequents}}$ and S-sequents and is defined as follows:

(Def. 36) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1\rangle \in \mathrm{P3d}\,S$ iff s_1 rule 3d S_1.

The functor $\mathrm{P3e}\,S$ yielding a relation between $2^{S\text{-sequents}}$ and S-sequents is defined by:

(Def. 37) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1\rangle \in \mathrm{P3e}\,S$ iff s_1 rule 3e S_1.

The functor $\mathrm{P4}\,S$ yielding a relation between $2^{S\text{-sequents}}$ and S-sequents is defined by:

(Def. 38) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1\rangle \in \mathrm{P4}\,S$ iff s_1 rule 4 S_1.

The functor $\mathrm{P5}\,S$ yields a relation between $2^{S\text{-sequents}}$ and S-sequents and is defined by:

(Def. 39) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1\rangle \in \mathrm{P5}\,S$ iff s_1 rule 5 S_1.

The functor $\mathrm{P6}\,S$ yielding a relation between $2^{S\text{-sequents}}$ and S-sequents is defined by:

(Def. 40) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1\rangle \in \mathrm{P6}\,S$ iff s_1 rule 6 S_1.

The functor $\mathrm{P7}\,S$ yielding a relation between $2^{S\text{-sequents}}$ and S-sequents is defined as follows:

(Def. 41) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1\rangle \in \mathrm{P7}\,S$ iff s_1 rule 7 S_1.

The functor $\mathrm{P8}\,S$ yields a relation between $2^{S\text{-sequents}}$ and S-sequents and is defined as follows:

(Def. 42) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1\rangle \in \mathrm{P8}\,S$ iff s_1 rule 8 S_1.

The functor $\mathrm{P9}\,S$ yields a relation between $2^{S\text{-sequents}}$ and S-sequents and is defined as follows:

(Def. 43) For every element S_1 of $2^{S\text{-sequents}}$ and for every element s_1 of S-sequents holds $\langle S_1, s_1\rangle \in \mathrm{P9}\,S$ iff s_1 rule 9 S_1.

Let us consider S and let R be a relation between $2^{S\text{-sequents}}$ and S-sequents. The functor FuncRule R yields a rule of S and is defined by:

(Def. 44) For every set i_1 such that $i_1 \in 2^{S\text{-sequents}}$ holds $(\mathrm{FuncRule}\, R)(i_1) = \{x \in S\text{-sequents}: \langle i_1, x\rangle \in R\}$.

Let us consider S. The functor R0 S yielding a rule of S is defined as follows:

(Def. 45) $\mathrm{R0}\, S = \mathrm{FuncRule}\, \mathrm{P0}\, S$.

The functor R1 S yielding a rule of S is defined as follows:

(Def. 46) $\mathrm{R1}\, S = \mathrm{FuncRule}\, \mathrm{P1}\, S$.

The functor R2 S yielding a rule of S is defined by:

(Def. 47) $\mathrm{R2}\, S = \mathrm{FuncRule}\, \mathrm{P2}\, S$.

The functor R3a S yielding a rule of S is defined by:

(Def. 48) $\mathrm{R3a}\, S = \mathrm{FuncRule}\, \mathrm{P3a}\, S$.

The functor R3b S yielding a rule of S is defined as follows:

(Def. 49) $\mathrm{R3b}\, S = \mathrm{FuncRule}\, \mathrm{P3b}\, S$.

The functor R3d S yielding a rule of S is defined as follows:

(Def. 50) $\mathrm{R3d}\, S = \mathrm{FuncRule}\, \mathrm{P3d}\, S$.

The functor R3e S yielding a rule of S is defined by:

(Def. 51) $\mathrm{R3e}\, S = \mathrm{FuncRule}\, \mathrm{P3e}\, S$.

The functor R4 S yields a rule of S and is defined as follows:

(Def. 52) $\mathrm{R4}\, S = \mathrm{FuncRule}\, \mathrm{P4}\, S$.

The functor R5 S yielding a rule of S is defined as follows:

(Def. 53) $\mathrm{R5}\, S = \mathrm{FuncRule}\, \mathrm{P5}\, S$.

The functor R6 S yields a rule of S and is defined by:

(Def. 54) $\mathrm{R6}\, S = \mathrm{FuncRule}\, \mathrm{P6}\, S$.

The functor R7 S yields a rule of S and is defined by:

(Def. 55) $\mathrm{R7}\, S = \mathrm{FuncRule}\, \mathrm{P7}\, S$.

The functor R8 S yielding a rule of S is defined as follows:

(Def. 56) $\mathrm{R8}\, S = \mathrm{FuncRule}\, \mathrm{P8}\, S$.

The functor R9 S yields a rule of S and is defined by:

(Def. 57) $\mathrm{R9}\, S = \mathrm{FuncRule}\, \mathrm{P9}\, S$.

Let us consider S and let t be a termal string of S.

Note that $\{\langle\emptyset, \langle\mathrm{TheEqSymbOf}\, S\rangle \frown t \frown t\rangle\}$ is $\{\mathrm{R2}\, S\}$-derivable. Note that R2 S is isotone. One can verify that R3b S is isotone.

Let t, t_1, t_2 be termal strings of S, and let p_5 be a finite subset of AllFormulasOf S. Observe that $\langle p_5 \cup \{\langle\mathrm{TheEqSymbOf}\, S\rangle \frown t_1 \frown t_2\}, \langle\mathrm{TheEqSymbOf}\, S\rangle \frown t \frown t_2\rangle$ is $(1, \{\langle p_5, \langle\mathrm{TheEqSymbOf}\, S\rangle \frown t \frown t_1\rangle\}, \{\mathrm{R3a}\, S\})$-derivable.

Let us consider S, let t, t_1, t_2 be termal strings of S, and let p_2 be a w.f.f. string of S. Note that $\langle\{p_2, \langle \mathrm{TheEqSymbOf}\, S\rangle \frown t_1 \frown t_2\}, \langle \mathrm{TheEqSymbOf}\, S\rangle \frown t \frown t_2\rangle$ is $(1, \{\langle\{p_2\}, \langle \mathrm{TheEqSymbOf}\, S\rangle \frown t \frown t_1\rangle\}, \{\mathrm{R3a}\, S\})$-derivable.

Let us consider S, let p_2 be a w.f.f. string of S, and let P_1 be a finite subset of AllFormulasOf S. One can verify that $\langle P_1 \cup \{p_2\}, p_2\rangle$ is $(1, \emptyset, \{\mathrm{R0}\, S\})$-derivable.

Let us consider S and let p_3, p_4 be w.f.f. strings of S. One can check that $\langle\{p_3, p_4\}, p_3\rangle$ is $(1, \emptyset, \{\mathrm{R0}\, S\})$-derivable.

Let us consider S, p_2. Note that $\langle\{p_2\}, p_2\rangle$ is $(1, \emptyset, \{\mathrm{R0}\, S\})$-derivable.

Let us consider S and let p_2 be a w.f.f. string of S. Observe that $\{\langle\{p_2\}, p_2\rangle\}$ is $(\emptyset, \{\mathrm{R0}\, S\})$-derivable.

Let us consider S. One can verify the following observations:

* R0 S is isotone,
* R3a S is isotone,
* R3d S is isotone, and
* R3e S is isotone.

Let us consider K_1, K_2. One can verify that $K_1 \cup K_2$ is isotone.

Let us consider S and let t_1, t_2 be termal strings of S.

Observe that $\langle \mathrm{TheEqSymbOf}\, S\rangle \frown t_1 \frown t_2$ is 0-w.f.f..

Let us consider S, let m be a non zero natural number, and let T, U be m-element elements of $(\mathrm{AllTermsOf}\, S)^*$. The functor PairWiseEq$(T, U)$ is defined by the condition (Def. 58).

(Def. 58) $\mathrm{PairWiseEq}(T, U) = \{\langle \mathrm{TheEqSymbOf}\, S\rangle \frown T_1(j) \frown U_1(j); j$ ranges over elements of $\mathrm{Seg}\, m, T_1$ ranges over functions from $\mathrm{Seg}\, m$ into $(\mathrm{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}, U_1$ ranges over functions from $\mathrm{Seg}\, m$ into $(\mathrm{AllSymbolsOf}\, S)^* \setminus \{\emptyset\} : T_1 = T \ \wedge\ U_1 = U\}$.

Let us consider S, let m be a non zero natural number, and let T_2, T_3 be m-element elements of $(\mathrm{AllTermsOf}\, S)^*$. Then PairWiseEq$(T_2, T_3)$ is a subset of AllFormulasOf S.

Let us consider S, let m be a non zero natural number, and let T, U be m-element elements of $(\mathrm{AllTermsOf}\, S)^*$. Observe that PairWiseEq(T, U) is finite.

Let us consider S, let s be a relational element of S, and let T_2, T_3 be $|\mathrm{ar}\, s|$-element elements of $(\mathrm{AllTermsOf}\, S)^*$. Observe that $\{\langle \mathrm{PairWiseEq}(T_2, T_3) \cup \{s\text{-compound}\, T_2\}, s\text{-compound}\, T_3\rangle\}$ is $(\emptyset, \{\mathrm{R3e}\, S\})$-derivable.

Let us consider m, S, D. We say that D is m-ranked if and only if:

(Def. 59)(i) R0 S, R2 S, R3a S, R3b $S \in D$ if $m = 0$,

(ii) R0 S, R2 S, R3a S, R3b S, R3d S, R3e $S \in D$ if $m = 1$,

(iii) R0 S, R1 S, R2 S, R3a S, R3b S, R3d S, R3e S, R4 S, R5 S, R6 S, R7 S, R8 $S \in D$ if $m = 2$,

(iv) $D = \emptyset$, otherwise.

Let us consider S. One can verify that every rule set of S which is 1-ranked is also 0-ranked and every rule set of S which is 2-ranked is also 1-ranked.

Let us consider S. The functor S-rules yields a rule set of S and is defined by:

(Def. 60) $S\text{-rules} = \{\mathrm{R0}\,S, \mathrm{R1}\,S, \mathrm{R2}\,S, \mathrm{R3a}\,S, \mathrm{R3b}\,S, \mathrm{R3d}\,S, \mathrm{R3e}\,S, \mathrm{R4}\,S\} \cup \{\mathrm{R5}\,S, \mathrm{R6}\,S, \mathrm{R7}\,S, \mathrm{R8}\,S\}$.

Let us consider S. Observe that S-rules is 2-ranked.

Let us consider S. Note that there exists a rule set of S which is 2-ranked.

Let us consider S. Observe that there exists a rule set of S which is 1-ranked.

Let us consider S. Note that there exists a rule set of S which is 0-ranked.

Let us consider S, let D be a 1-ranked rule set of S, let X be a D-expanded set, and let us consider a. Observe that X-freeInterpreter a is (X, D)-termEq-respecting.

Let us consider S, let D be a 0-ranked rule set of S, and let X be a D-expanded set. Observe that (X, D)-termEq is total, symmetric, and transitive.

Let us consider S. Observe that there exists a 0-ranked rule set of S which is 1-ranked.

The following proposition is true

(1) If $D_1 \subseteq D_2$ and if D_2 is isotone or D_1 is isotone and if Y is (X, D_1)-derivable, then Y is (X, D_2)-derivable.

Let us consider S, S_6. One can verify that $\{S_6\}$ is S-sequents-like.

Let us consider S, S_{11}, S_5. One can check that $S_{11} \cup S_5$ is S-sequents-like.

Let us consider S and let x, y be S-sequent-like sets. Observe that $\{x, y\}$ is S-sequents-like.

Let us consider S, p_3, p_4. Note that $\langle\{\mathrm{xnot}\,p_3, \mathrm{xnot}\,p_4\}, \langle\mathrm{TheNorSymbOf}\,S\rangle ^\frown p_3 ^\frown p_4\rangle$ is $(1, \{\langle\{\mathrm{xnot}\,p_3, \mathrm{xnot}\,p_4\}, \mathrm{xnot}\,p_3\rangle, \langle\{\mathrm{xnot}\,p_3, \mathrm{xnot}\,p_4\}, \mathrm{xnot}\,p_4\rangle\}, \{\mathrm{R6}\,S\})$-derivable.

Let us consider S, p_3, p_4. One can check that $\langle\{p_3, p_4\}, p_4\rangle$ is $(1, \emptyset, \{\mathrm{R0}\,S\})$-derivable.

We now state two propositions:

(2) For every relation R between $2^{S\text{-sequents}}$ and S-sequents such that $\langle S_4, S_6\rangle \in R$ holds $S_6 \in (\mathrm{FuncRule}\,R)(S_4)$.

(3) If $x \in R(X)$, then x is $(1, X, \{R\})$-derivable.

Let us consider S, D, X. Let us observe that X is D-expanded if and only if:

(Def. 61) If x is (X, D)-provable, then $x \in X$.

The following four propositions are true:

(4) If $p_2 \in X$, then p_2 is $(X, \{\mathrm{R0}\,S\})$-provable.

(5) Suppose that

(i) $D_1 \cup D_2$ is isotone,

(ii) $D_1 \cup D_2 \cup D_3$ is isotone,
(iii) x is (m, S_{11}, D_1)-derivable,
(iv) y is (m, S_5, D_2)-derivable, and
(v) z is $(n, \{x, y\}, D_3)$-derivable.
Then z is $(m + n, S_{11} \cup S_5, D_1 \cup D_2 \cup D_3)$-derivable.

(6) Suppose D_1 is isotone and $D_1 \cup D_2$ is isotone and y is (m, X, D_1)-derivable and z is $(n, \{y\}, D_2)$-derivable. Then z is $(m + n, X, D_1 \cup D_2)$-derivable.

(7) If x is (m, X, D)-derivable, then $\{x\}$ is (X, D)-derivable.

Let us consider S. Observe that R6 S is isotone.

One can prove the following propositions:

(8) If $D_1 \subseteq D_2$ and if D_1 is isotone or D_2 is isotone and if x is (X, D_1)-provable, then x is (X, D_2)-provable.

(9) If $X \subseteq Y$ and x is (X, D)-provable, then x is (Y, D)-provable.

Let us consider S. Note that R8 S is isotone.

Let us consider S. Observe that R1 S is isotone.

Next we state the proposition

(10) If $\{y\}$ is (S_4, D)-derivable, then there exists m_1 such that y is (m_1, S_4, D)-derivable.

Let us consider S, D, X. Observe that every set which is (X, D)-derivable is also S-sequents-like.

Let us consider S, D, X, x. Let us observe that x is (X, D)-provable if and only if:

(Def. 62) There exists a set H and there exists m such that $H \subseteq X$ and $\langle H, x \rangle$ is $(m, \emptyset, D)$-derivable.

The following proposition is true

(11) If $D_1 \subseteq D_2$ and if D_2 is isotone or D_1 is isotone and if x is (m, X, D_1)-derivable, then x is (m, X, D_2)-derivable.

Let us consider S. Observe that R7 S is isotone.

Next we state the proposition

(12) If x is (X, D)-provable, then x is a w.f.f. string of S.

In the sequel F denotes a rule set of S.

Let us consider S, D_1 and let X be a D_1-expanded set. One can verify that (S, X)-freeInterpreter is (X, D_1)-termEq-respecting.

Let us consider S, let D be a 0-ranked rule set of S, and let X be a D-expanded set. The functor D Henkin X yielding a function is defined by:

(Def. 63) D Henkin $X = (S, X)$-freeInterpreter quotient(X, D)-termEq.

Let us consider S, let D be a 0-ranked rule set of S, and let X be a D-expanded set. One can check that D Henkin X is OwnSymbolsOf S-defined.

Let us consider S, D_1 and let X be a D_1-expanded set. Observe that D_1 Henkin X is $(S, \mathrm{Classes}(X, D_1)$ -termEq)-interpreter-like.

Let us consider S, D_1 and let X be a D_1-expanded set. Then D_1 Henkin X is an element of Classes((X, D_1) -termEq)-InterpretersOf S.

Let us consider S, p_3, p_4. One can verify that $\langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_3 \frown p_4$ is $(\{\mathrm{xnot}\, p_3, \mathrm{xnot}\, p_4\}, \{\mathrm{R0}\, S\} \cup \{\mathrm{R6}\, S\})$-provable.

Let us consider S. Note that every 0-ranked rule set of S is non empty.

Let us consider S, x. We say that x is S-premises-like if and only if:

(Def. 64) $x \subseteq \mathrm{AllFormulasOf}\, S$ and x is finite.

Let us consider S. One can verify that every set which is S-premises-like is also finite.

Let us consider S, p_2. Note that $\{p_2\}$ is S-premises-like.

Let us consider S and let e be an empty set. One can check that e null S is S-premises-like.

Let us consider X, S. Observe that there exists a subset of X which is S-premises-like.

Let us consider S. Observe that there exists a set which is S-premises-like.

Let us consider S and let X be an S-premises-like set. Observe that every subset of X is S-premises-like.

In the sequel H_3 denotes an S-premises-like set.

Let us consider S, H_2, H_1. Then H_1 null H_2 is a subset of AllFormulasOf S.

Let us consider S, H, x. Note that H null x is S-premises-like.

Let us consider S, H_1, H_2. Note that $H_1 \cup H_2$ is S-premises-like.

Let us consider S, H, p_2. Observe that $\langle H, p_2\rangle$ is S-sequent-like.

Let us consider S, H_1, H_2, p_2. One can verify that $\langle H_1 \cup H_2, p_2\rangle$ is $(1, \{\langle H_1, p_2\rangle\}, \{\mathrm{R1}\, S\})$-derivable.

Let us consider S, H, p_2, p_3, p_4. One can check that $\langle H\, \mathrm{null}\, p_3 \frown p_4, \mathrm{xnot}\, p_2\rangle$ is $(1, \{\langle H \cup \{p_2\}, p_3\rangle, \langle H \cup \{p_2\}, \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_3 \frown p_4\rangle\}, \{\mathrm{R8}\, S\})$-derivable.

Let us consider S. One can verify that $\emptyset$ null S is S-sequents-like.

Let us consider S, H, p_2. Observe that $\langle H \cup \{p_2\}, p_2\rangle$ is $(1, \emptyset, \{\mathrm{R0}\, S\})$-derivable. Let us consider p_3, p_4. Note that $\langle H\, \mathrm{null}\, p_4, \mathrm{xnot}\, p_3\rangle$ is

$(2, \{\langle H, \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_3 \frown p_4\rangle\}, \{\mathrm{R0}\, S\} \cup \{\mathrm{R1}\, S\} \cup \{\mathrm{R8}\, S\})$-derivable.

Let us consider S, H, p_3, p_4. Note that $\langle H, \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_4 \frown p_3\rangle$ is $(1, \{\langle H, \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_3 \frown p_4\rangle\}, \{\mathrm{R7}\, S\})$-derivable.

Let us consider S, H, p_3, p_4. Observe that $\langle H\, \mathrm{null}\, p_3, \mathrm{xnot}\, p_4\rangle$ is $(3, \{\langle H, \langle \mathrm{TheNorSymbOf}\, S\rangle \frown p_3 \frown p_4\rangle\}, \{\mathrm{R0}\, S\} \cup \{\mathrm{R1}\, S\} \cup \{\mathrm{R8}\, S\} \cup \{\mathrm{R7}\, S\})$-derivable.

Let us consider S, S_6. Observe that $(S_6)_{\mathbf{1}}$ is S-premises-like.

Let us consider S, X, D. Then D null X is a rule set of S.

Let us consider S, p_3, p_4, l_1, H and let l_2 be an $H \cup \{p_3\} \cup \{p_4\}$-absent literal element of S.

Note that $\langle (H \cup \{\langle l_1 \rangle \frown p_3\}) \operatorname{null} l_2,\ p_4 \rangle$ is $(1, \{\langle H \cup \{(l_1, l_2)\text{-SymbolSubstIn}\, p_3\}, p_4 \rangle\}, \{\text{R5}\, S\})$-derivable.

Let us consider S, D, X. We say that X is D-inconsistent if and only if:

(Def. 65) There exist p_3, p_4 such that p_3 is (X, D)-provable and $\langle \text{TheNorSymbOf}\, S \rangle \frown p_3 \frown p_4$ is (X, D)-provable.

Let us consider m_2, S, H_1, H_2, p_2. Note that $\langle (H_1 \cup H_2) \operatorname{null} m_2,\ p_2 \rangle$ is $(m_2, \{\langle H_1, p_2 \rangle\}, \{\text{R1}\, S\})$-derivable.

Let us consider S. Observe that there exists an isotone rule set of S which is non empty.

We now state the proposition

(13) If X is D-inconsistent and D is isotone and $\text{R1}\, S, \text{R8}\, S \in D$, then $\text{xnot}\, p_2$ is (X, D)-provable.

Let us consider S. Observe that $\text{R5}\, S$ is isotone.

Let us consider S, l, t, p_2. Observe that $\langle \{(l, t) \operatorname{SubstIn} p_2\},\ \langle l \rangle \frown p_2 \rangle$ is $(1, \emptyset, \{\text{R4}\, S\})$-derivable.

Let us consider S. One can verify that $\text{R4}\, S$ is isotone.

Let us consider S, X. We say that X is S-witnessed if and only if:

(Def. 66) For all l_1, p_3 such that $\langle l_1 \rangle \frown p_3 \in X$ there exists l_2 such that $(l_1, l_2)\text{-SymbolSubstIn}\, p_3 \in X$ and $l_2 \notin \operatorname{rng} p_3$.

We now state the proposition

(14)[3] Let X be a D_1-expanded set. Suppose $\text{R1}\, S$, $\text{R4}\, S$, $\text{R6}\, S$, $\text{R7}\, S$, $\text{R8}\, S \in D_1$ and X is S-mincover and S-witnessed. Then $(D_1 \operatorname{Henkin} X)\text{-TruthEval}\, p_1 = 1$ if and only if $p_1 \in X$.

Let us consider S, D, X. We introduce X is D-consistent as an antonym of X is D-inconsistent.

We now state the proposition

(15) For every subset X of Y such that X is D-inconsistent holds Y is D-inconsistent.

Let us consider S, D, let X be a functional set, and let p_2 be an element of $\text{ExFormulasOf}\, S$. The functor $(D, p_2) \operatorname{AddAsWitnessTo} X$ is defined by:

(Def. 67)
$$(D, p_2) \operatorname{AddAsWitnessTo} X = \begin{cases} X \cup \{(S\text{-firstChar}(p_2),\ \text{the element of LettersOf}\, S \setminus \text{SymbolsOf}(((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}) \cap (X \cup \{\operatorname{head} p_2\})))\text{-SymbolSubstIn} \operatorname{head} p_2\}, \\ \quad \text{if } X \cup \{p_2\} \text{ is } D\text{-consistent and LettersOf}\, S \setminus \text{SymbolsOf}(((\text{AllSymbolsOf}\, S)^* \setminus \{\emptyset\}) \cap (X \cup \{\operatorname{head} p_2\})) \neq \emptyset, \\ X, \text{ otherwise.} \end{cases}$$

[3] Henkin's Theorem

Let us consider S, D, let X be a functional set, and let p_2 be an element of ExFormulasOf S. One can check that $X \setminus ((D, p_2)$ AddAsWitnessTo $X)$ is empty.

Let us consider S, D, let X be a functional set, and let p_2 be an element of ExFormulasOf S. One can check that $((D, p_2)$ AddAsWitnessTo $X) \setminus X$ is trivial.

Let us consider S, D, let X be a functional set, and let p_2 be an element of ExFormulasOf S. Then (D, p_2) AddAsWitnessTo X is a subset of $X \cup$ AllFormulasOf S.

Let us consider S, D. We say that D is correct if and only if the condition (Def. 68) is satisfied.

(Def. 68) Let given p_2, X. Suppose p_2 is (X, D)-provable. Let given U and I be an element of U-InterpretersOf S. If X is I-satisfied, then I-TruthEval $p_2 = 1$.

Let us consider S, t_1, t_2. One can check that SubTerms($\langle$TheEqSymbOf $S\rangle \frown t_1 \frown t_2) \dot{-} \langle t_1, t_2 \rangle$ is empty.

Let us consider S and let R be a rule of S. We say that R is correct if and only if:

(Def. 69) If X is S-correct, then $R(X)$ is S-correct.

Let us consider S. Observe that every set which is S-sequent-like is also S-null.

Let us consider S. Note that R0 S is correct.

Let us consider S. Note that there exists a rule of S which is correct.

Let us consider S. One can check that R1 S is correct.

Let us consider S. Note that R2 S is correct.

Let us consider S. One can check that R3a S is correct.

Let us consider S. Observe that R3b S is correct.

Let us consider S. Observe that R3d S is correct.

Let us consider S. Note that R3e S is correct.

Let us consider S. One can check that R4 S is correct.

Let us consider S. One can check that R5 S is correct.

Let us consider S. One can verify that R6 S is correct.

Let us consider S. Observe that R7 S is correct.

Let us consider S. Observe that R8 S is correct.

Next we state the proposition

(16) If for every rule R of S such that $R \in D$ holds R is correct, then D is correct.

Let us consider S and let R be a correct rule of S. Note that $\{R\}$ is correct. Observe that S-rules is correct. One can check that R9 S is isotone. Let us consider H, p_2. Observe that $\langle H, p_2 \rangle$ null 1 is $(1, \{\langle H, \text{xnot xnot } p_2 \rangle\}, \{\text{R9 } S\})$-derivable.

Let us consider X, S. Observe that there exists an 0-w.f.f. string of S which is X-implied.

Let us consider X, S. Observe that there exists a w.f.f. string of S which is X-implied.

Let us consider S, X and let p_2 be an X-implied w.f.f. string of S. Observe that xnot xnot p_2 is X-implied.

Let us consider X, S, p_2. We say that p_2 is X-provable if and only if:

(Def. 70) p_2 is $(X, \{\mathrm{R9}\,S\} \cup S\text{-rules})$-provable.

2. Constructions for Countable Languages: Witness Adjoining

Let X be a functional set, let us consider S, D, and let n_1 be a function from $\mathbb{N}$ into ExFormulasOf S. The functor (D, n_1) AddWitnessesTo X yields a function from $\mathbb{N}$ into $2^{X \cup \mathrm{AllFormulasOf}\,S}$ and is defined by:

(Def. 71) $((D, n_1)\,\mathrm{AddWitnessesTo}\,X)(0) = X$ and for every m_1 holds $((D, n_1)\,\mathrm{AddWitnessesTo}\,X)(m_1 + 1) = (D, n_1(m_1))\,\mathrm{AddAsWitnessTo}((D, n_1)\,\mathrm{AddWitnessesTo}\,X)(m_1)$.

Let X be a functional set, let us consider S, D, and let n_1 be a function from $\mathbb{N}$ into ExFormulasOf S. We introduce (D, n_1) addw X as a synonym of (D, n_1) AddWitnessesTo X.

We now state the proposition

(17) Let X be a functional set and n_1 be a function from $\mathbb{N}$ into ExFormulasOf S. Suppose D is isotone and R1 S, R8 S, R2 S, R5 $S \in D$ and LettersOf $S \setminus$ SymbolsOf$(X \cap ((\mathrm{AllSymbolsOf}\,S)^* \setminus \{\emptyset\}))$ is infinite and X is D-consistent. Then $((D, n_1)\,\mathrm{addw}\,X)(k) \subseteq ((D, n_1)\,\mathrm{addw}\,X)(k + m)$ and LettersOf $S \setminus$ SymbolsOf$(((D, n_1)\,\mathrm{addw}\,X)(m) \cap ((\mathrm{AllSymbolsOf}\,S)^* \setminus \{\emptyset\}))$ is infinite and $((D, n_1)\,\mathrm{addw}\,X)(m)$ is D-consistent.

Let X be a functional set, let us consider S, D, and let n_1 be a function from $\mathbb{N}$ into ExFormulasOf S. The functor X WithWitnessesFrom(D, n_1) yielding a subset of $X \cup$ AllFormulasOf S is defined by:

(Def. 72) $X\,\mathrm{WithWitnessesFrom}(D, n_1) = \bigcup \mathrm{rng}((D, n_1)\,\mathrm{AddWitnessesTo}\,X)$.

Let X be a functional set, let us consider S, D, and let n_1 be a function from $\mathbb{N}$ into ExFormulasOf S. We introduce X addw(D, n_1) as a synonym of X WithWitnessesFrom(D, n_1).

Let X be a functional set, let us consider S, D, and let n_1 be a function from $\mathbb{N}$ into ExFormulasOf S. One can verify that $X \setminus (X\,\mathrm{addw}(D, n_1))$ is empty.

The following proposition is true

(18) Let X be a functional set and n_1 be a function from $\mathbb{N}$ into ExFormulasOf S. Suppose that D is isotone and R1 S, R8 S, R2 S, R5 $S \in D$ and LettersOf $S \setminus$ SymbolsOf$(X \cap ((\mathrm{AllSymbolsOf}\,S)^* \setminus \{\emptyset\}))$ is infinite and $X\,\mathrm{addw}(D, n_1) \subseteq Z$ and Z is D-consistent and $\mathrm{rng}\,n_1 =$ ExFormulasOf S. Then Z is S-witnessed.

3. Constructions for Countable Languages: Consistently Maximizing a Set of Formulas of a Countable Language (Lindenbaum's Lemma)

Let us consider X, S, D and let p_2 be an element of AllFormulasOf S. The functor (D, p_2) AddFormulaTo X is defined by:

(Def. 73) $(D, p_2)\,\mathrm{AddFormulaTo}\,X = \begin{cases} X \cup \{p_2\}, & \\ \quad \text{if } \mathrm{xnot}\,p_2 \text{ is not } (X, D)\text{-provable}, & \\ X \cup \{\mathrm{xnot}\,p_2\}, \text{ otherwise.} & \end{cases}$

Let us consider X, S, D and let p_2 be an element of AllFormulasOf S. Then (D, p_2) AddFormulaTo X is a subset of $X \cup$ AllFormulasOf S.

Let us consider X, S, D and let p_2 be an element of AllFormulasOf S. Note that $X \setminus ((D, p_2)\,\mathrm{AddFormulaTo}\,X)$ is empty.

Let us consider X, S, D and let n_1 be a function from $\mathbb{N}$ into AllFormulasOf S. The functor (D, n_1) AddFormulasTo X yields a function from $\mathbb{N}$ into

$2^{X \cup \mathrm{AllFormulasOf}\,S}$ and is defined by:

(Def. 74) $((D, n_1)\,\mathrm{AddFormulasTo}\,X)(0) = X$ and for every m holds $((D, n_1)\,\mathrm{AddFormulasTo}\,X)(m + 1) =$ $(D, n_1(m))\,\mathrm{AddFormulaTo}((D, n_1)\,\mathrm{AddFormulasTo}\,X)(m)$.

Let us consider X, S, D and let n_1 be a function from $\mathbb{N}$ into AllFormulasOf S. The functor (D, n_1) CompletionOf X yields a subset of $X \cup$ AllFormulasOf S and is defined as follows:

(Def. 75) $(D, n_1)\,\mathrm{CompletionOf}\,X = \bigcup \mathrm{rng}((D, n_1)\,\mathrm{AddFormulasTo}\,X)$.

Let us consider X, S, D and let n_1 be a function from $\mathbb{N}$ into AllFormulasOf S. One can check that $X \setminus ((D, n_1)\,\mathrm{CompletionOf}\,X)$ is empty.

We now state the proposition

(19) For every relation R between $2^{S\text{-sequents}}$ and S-sequents holds $y \in (\mathrm{FuncRule}\,R)(X)$ iff $y \in S$-sequents and $\langle X, y \rangle \in R$.

In the sequel D_2 is a 2-ranked rule set of S.

Let us consider S and let r_1, r_2 be isotone rules of S. Note that $\{r_1, r_2\}$ is isotone.

Let us consider S and let r_1, r_2, r_3, r_4 be isotone rules of S. Observe that $\{r_1, r_2, r_3, r_4\}$ is isotone.

Let us consider S. Observe that S-rules is isotone.

Let us consider S. Observe that there exists an isotone rule set of S which is correct.

Let us consider S. Observe that there exists a correct isotone rule set of S which is 2-ranked.

Let S be a countable language. Observe that AllFormulasOf S is countable.

We now state the proposition

(20) Let S be a countable language and D be a rule set of S. Suppose D is 2-ranked, isotone, and correct and Z is D-consistent and $Z \subseteq$ AllFormulasOf S. Then there exists a non empty set U and there exists an element I of U-InterpretersOf S such that Z is I-satisfied.

In the sequel C denotes a countable language and p_2 denotes a w.f.f. string of C.

We now state the proposition

(21) If $X \subseteq$ AllFormulasOf C and p_2 is X-implied, then p_2 is X-provable.

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