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Partial Differentiation, Differentiation and Continuity on n-Dimensional Real Normed Linear Spaces

Takao Inoué
Inaba 2205, Wing-Minamikan
Nagano, Nagano, Japan

Adam Naumowicz
Institute of Computer Science
University of Białystok
Akademicka 2, 15-267 Białystok, Poland

Noboru Endou
Nagano National College of Technology
Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, we aim to prove the characterization of differentiation by means of partial differentiation for vector-valued functions on n-dimensional real normed linear spaces (refer to [15] and [16]).

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The notation and terminology used in this paper have been introduced in the following papers: [2], [7], [1], [3], [4], [5], [17], [11], [13], [6], [9], [14], [10], [8], [12], and [18].

One can prove the following propositions:

(1) Let n, i be elements of $\mathbb{N}$, q be an element of $\mathcal{R}^n$, and p be a point of $\mathcal{E}^n_{\mathrm{T}}$. If $i \in \operatorname{Seg} n$ and $q = p$, then $|p_i| \leq |q|$.

(2) For every real number x and for every element v_1 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ such that $v_1 = \langle x \rangle$ holds $\|v_1\| = |x|$.

(3) Let n be a non empty element of $\mathbb{N}$, x be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and i be an element of $\mathbb{N}$. If $1 \leq i \leq n$, then $\|(\operatorname{Proj}(i, n))(x)\| \leq \|x\|$.

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(4) For every non empty element n of $\mathbb{N}$ and for every element x of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and for every element i of $\mathbb{N}$ holds $\|(\mathrm{Proj}(i,n))(x)\| = |(\mathrm{proj}(i,n))(x)|$.

(5) Let n be a non empty element of $\mathbb{N}$, x be an element of $\mathcal{R}^n$, and i be an element of $\mathbb{N}$. If $1 \leq i \leq n$, then $|(\mathrm{proj}(i,n))(x)| \leq |x|$.

(6) Let m, n be non empty elements of $\mathbb{N}$, s be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and i be an element of $\mathbb{N}$. Suppose $1 \leq i \leq n$. Then $\mathrm{Proj}(i,n)$ is a bounded linear operator from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and $(\mathrm{BdLinOpsNorm}(\langle \mathcal{E}^n, \|\cdot\| \rangle, \langle \mathcal{E}^1, \|\cdot\| \rangle))(\mathrm{Proj}(i,n)) \leq 1$.

(7) Let m, n be non empty elements of $\mathbb{N}$, s be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and i be an element of $\mathbb{N}$. Suppose $1 \leq i \leq n$. Then

(i) $\mathrm{Proj}(i,n) \cdot s$ is a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and

(ii) $(\mathrm{BdLinOpsNorm}(\langle \mathcal{E}^m, \|\cdot\| \rangle, \langle \mathcal{E}^1, \|\cdot\| \rangle))(\mathrm{Proj}(i,n) \cdot s) \leq (\mathrm{BdLinOpsNorm}(\langle \mathcal{E}^n, \|\cdot\| \rangle, \langle \mathcal{E}^1, \|\cdot\| \rangle))(\mathrm{Proj}(i,n)) \cdot (\mathrm{BdLinOpsNorm}(\langle \mathcal{E}^m, \|\cdot\| \rangle, \langle \mathcal{E}^n, \|\cdot\| \rangle))(s)$.

(8) For every non empty element n of $\mathbb{N}$ and for every element i of $\mathbb{N}$ holds $\mathrm{Proj}(i,n)$ is homogeneous.

(9) Let n be a non empty element of $\mathbb{N}$, x be an element of $\mathcal{R}^n$, r be a real number, and i be an element of $\mathbb{N}$. Then $(\mathrm{proj}(i,n))(r \cdot x) = r \cdot (\mathrm{proj}(i,n))(x)$.

(10) Let n be a non empty element of $\mathbb{N}$, x, y be elements of $\mathcal{R}^n$, and i be an element of $\mathbb{N}$. Then $(\mathrm{proj}(i,n))(x+y) = (\mathrm{proj}(i,n))(x) + (\mathrm{proj}(i,n))(y)$.

(11) Let n be a non empty element of $\mathbb{N}$, x, y be points of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and i be an element of $\mathbb{N}$. Then $(\mathrm{Proj}(i,n))(x-y) = (\mathrm{Proj}(i,n))(x) - (\mathrm{Proj}(i,n))(y)$.

(12) Let n be a non empty element of $\mathbb{N}$, x, y be elements of $\mathcal{R}^n$, and i be an element of $\mathbb{N}$. Then $(\mathrm{proj}(i,n))(x-y) = (\mathrm{proj}(i,n))(x) - (\mathrm{proj}(i,n))(y)$.

(13) Let m, n be non empty elements of $\mathbb{N}$, s be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, i be an element of $\mathbb{N}$, and s_1 be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$. If $s_1 = \mathrm{Proj}(i,n) \cdot s$ and $1 \leq i \leq n$, then $\|s_1\| \leq \|s\|$.

(14) Let m, n be non empty elements of $\mathbb{N}$, s, t be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, s_1, t_1 be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and i be an element of $\mathbb{N}$. If $s_1 = \mathrm{Proj}(i,n) \cdot s$ and $t_1 = \mathrm{Proj}(i,n) \cdot t$ and $1 \leq i \leq n$, then $\|s_1 - t_1\| \leq \|s - t\|$.

(15) Let K be a real number, n be an element of $\mathbb{N}$, and s be an element of $\mathcal{R}^n$. Suppose that for every element i of $\mathbb{N}$ such that $1 \leq i \leq n$ holds

$|s(i)| \leq K$. Then $|s| \leq n \cdot K$.

(16) Let K be a real number, n be a non empty element of $\mathbb{N}$, and s be an element of $\langle \mathcal{E}^n, \| \cdot \| \rangle$. Suppose that for every element i of $\mathbb{N}$ such that $1 \leq i \leq n$ holds $\|(\mathrm{Proj}(i, n))(s)\| \leq K$. Then $\|s\| \leq n \cdot K$.

(17) Let K be a real number, n be a non empty element of $\mathbb{N}$, and s be an element of $\mathcal{R}^n$. Suppose that for every element i of $\mathbb{N}$ such that $1 \leq i \leq n$ holds $|(\mathrm{proj}(i, n))(s)| \leq K$. Then $|s| \leq n \cdot K$.

(18) Let m, n be non empty elements of $\mathbb{N}$, s be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and K be a real number. Suppose that for every element i of $\mathbb{N}$ and for every point s_1 of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $s_1 = \mathrm{Proj}(i, n) \cdot s$ and $1 \leq i \leq n$ holds $\|s_1\| \leq K$. Then $\|s\| \leq n \cdot K$.

(19) Let m, n be non empty elements of $\mathbb{N}$, s, t be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and K be a real number. Suppose that for every element i of $\mathbb{N}$ and for all points s_1, t_1 of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $s_1 = \mathrm{Proj}(i, n) \cdot s$ and $t_1 = \mathrm{Proj}(i, n) \cdot t$ and $1 \leq i \leq n$ holds $\|s_1 - t_1\| \leq K$. Then $\|s - t\| \leq n \cdot K$.

(20) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, X be a subset of $\langle \mathcal{E}^m, \| \cdot \| \rangle$, and i be an element of $\mathbb{N}$. Suppose $1 \leq i \leq m$ and X is open. Then the following statements are equivalent

(i) f is partially differentiable on X w.r.t. i and $f{\restriction}^i X$ is continuous on X,

(ii) for every element j of $\mathbb{N}$ such that $1 \leq j \leq n$ holds $\mathrm{Proj}(j, n) \cdot f$ is partially differentiable on X w.r.t. i and $\mathrm{Proj}(j, n) \cdot f{\restriction}^i X$ is continuous on X.

(21) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and X be a subset of $\langle \mathcal{E}^m, \| \cdot \| \rangle$. Suppose X is open. Then f is differentiable on X and $f'_{\restriction X}$ is continuous on X if and only if for every element j of $\mathbb{N}$ such that $1 \leq j \leq n$ holds $\mathrm{Proj}(j, n) \cdot f$ is differentiable on X and $(\mathrm{Proj}(j, n) \cdot f)'_{\restriction X}$ is continuous on X.

(22) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and X be a subset of $\langle \mathcal{E}^m, \| \cdot \| \rangle$. Suppose X is open. Then for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f{\restriction}^i X$ is continuous on X if and only if f is differentiable on X and $f'_{\restriction X}$ is continuous on X.

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