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Difference and Difference Quotient. Part IV

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Summary. In this article, we give some important theorems of forward difference, backward difference, central difference and difference quotient and forward difference, backward difference, central difference and difference quotient formulas of some special functions.

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The papers [2], [7], [13], [3], [1], [6], [9], [4], [14], [8], [5], [15], [11], [12], and [10] provide the notation and terminology for this paper.

We adopt the following rules: n denotes an element of $\mathbb{N}$, h, k, x, x_0, x_1, x_2, x_3 denote real numbers, and f, g denote functions from $\mathbb{R}$ into $\mathbb{R}$.

Next we state a number of propositions:

(1) If $x_0 > 0$ and $x_1 > 0$, then $\log_e x_0 - \log_e x_1 = \log_e(\frac{x_0}{x_1})$.

(2) If $x_0 > 0$ and $x_1 > 0$, then $\log_e x_0 + \log_e x_1 = \log_e(x_0 \cdot x_1)$.

(3) If $x > 0$, then $\log_e x =$ (the function ln)(x).

(4) If $x_0 > 0$ and $x_1 > 0$, then (the function ln)$(x_0)-$(the function ln)$(x_1) =$ (the function ln)$(\frac{x_0}{x_1})$.

(5) Suppose for every x holds $f(x) = \frac{k}{x^2}$ and $x_0 \neq 0$ and $x_1 \neq 0$ and $x_2 \neq 0$ and $x_3 \neq 0$ and x_0, x_1, x_2, x_3 are mutually different. Then $\Delta[f](x_0, x_1, x_2, x_3) = \frac{k \cdot (\frac{1}{x_1 \cdot x_2 \cdot x_0} \cdot (\frac{1}{x_0} + \frac{1}{x_2} + \frac{1}{x_1}) - \frac{1}{x_2 \cdot x_1 \cdot x_3} \cdot (\frac{1}{x_3} + \frac{1}{x_1} + \frac{1}{x_2}))}{x_0 - x_3}$.

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(6) Suppose $x_0 \in$ dom (the function cot) and $x_1 \in$ dom (the function cot). Then Δ[(the function cot) (the function cot)]$(x_0, x_1) = -\frac{(\cos x_1)^2 - (\cos x_0)^2}{(\sin x_0 \cdot \sin x_1)^2 \cdot (x_0 - x_1)}$.

(7) Suppose $x \in$ dom (the function cot) and $x + h \in$ dom (the function cot). Then $(\Delta_h$[(the function cot) (the function cot)]$)(x) = \frac{\frac{1}{2} \cdot (\cos(2 \cdot (x+h)) - \cos(2 \cdot x))}{(\sin(x+h) \cdot \sin x)^2}$.

(8) Suppose $x \in$ dom (the function cot) and $x - h \in$ dom (the function cot). Then $(\nabla_h$[(the function cot) (the function cot)]$)(x) = \frac{\frac{1}{2} \cdot (\cos(2 \cdot x) - \cos(2 \cdot (h-x)))}{(\sin x \cdot \sin(x-h))^2}$.

(9) Suppose $x + \frac{h}{2} \in$ dom (the function cot) and $x - \frac{h}{2} \in$ dom (the function cot). Then $(\delta_h$[(the function cot) (the function cot)]$)(x) = \frac{\frac{1}{2} \cdot (\cos(h+2 \cdot x) - \cos(h-2 \cdot x))}{(\sin(x+\frac{h}{2}) \cdot \sin(x-\frac{h}{2}))^2}$.

(10) If x_0, $x_1 \in \operatorname{dom} \operatorname{cosec}$, then
$\Delta[\operatorname{cosec} \ \operatorname{cosec}](x_0, x_1) = \frac{4 \cdot (\sin(x_1+x_0) \cdot \sin(x_1-x_0))}{(\cos(x_0+x_1) - \cos(x_0-x_1))^2 \cdot (x_0-x_1)}$.

(11) If x, $x + h \in \operatorname{dom} \operatorname{cosec}$, then $(\Delta_h[\operatorname{cosec} \ \operatorname{cosec}])(x) = -\frac{4 \cdot \sin(2 \cdot x+h) \cdot \sin h}{(\cos(2 \cdot x+h) - \cos h)^2}$.

(12) If x, $x - h \in \operatorname{dom} \operatorname{cosec}$, then $(\nabla_h[\operatorname{cosec} \ \operatorname{cosec}])(x) = -\frac{4 \cdot \sin(2 \cdot x-h) \cdot \sin h}{(\cos(2 \cdot x-h) - \cos h)^2}$.

(13) If $x+\frac{h}{2}$, $x-\frac{h}{2} \in \operatorname{dom} \operatorname{cosec}$, then $(\delta_h[\operatorname{cosec} \ \operatorname{cosec}])(x) = -\frac{4 \cdot \sin(2 \cdot x) \cdot \sin h}{(\cos(2 \cdot x) - \cos h)^2}$.

(14) If x_0, $x_1 \in \operatorname{dom} \sec$, then
$\Delta[\sec \ \sec](x_0, x_1) = \frac{4 \cdot (\sin(x_0+x_1) \cdot \sin(x_0-x_1))}{(\cos(x_0+x_1) + \cos(x_0-x_1))^2 \cdot (x_0-x_1)}$.

(15) If x, $x + h \in \operatorname{dom} \sec$, then $(\Delta_h[\sec \ \sec])(x) = \frac{4 \cdot \sin(2 \cdot x+h) \cdot \sin h}{(\cos(2 \cdot x+h) + \cos h)^2}$.

(16) If x, $x - h \in \operatorname{dom} \sec$, then $(\nabla_h[\sec \ \sec])(x) = \frac{4 \cdot \sin(2 \cdot x-h) \cdot \sin h}{(\cos(2 \cdot x-h) + \cos h)^2}$.

(17) If $x + \frac{h}{2}$, $x - \frac{h}{2} \in \operatorname{dom} \sec$, then $(\delta_h[\sec \ \sec])(x) = \frac{4 \cdot \sin(2 \cdot x) \cdot \sin h}{(\cos(2 \cdot x) + \cos h)^2}$.

(18) If x_0, $x_1 \in \operatorname{dom} \operatorname{cosec} \cap \operatorname{dom} \sec$, then $\Delta[\operatorname{cosec} \ \sec](x_0, x_1) = \frac{\frac{4 \cdot (\cos(x_1+x_0) \cdot \sin(x_1-x_0))}{\sin(2 \cdot x_0) \cdot \sin(2 \cdot x_1)}}{x_0-x_1}$.

(19) If $x + h$, $x \in \operatorname{dom} \operatorname{cosec} \cap \operatorname{dom} \sec$, then $(\Delta_h[\operatorname{cosec} \ \sec])(x) = -4 \cdot \frac{\cos(2 \cdot x+h) \cdot \sin h}{\sin(2 \cdot (x+h)) \cdot \sin(2 \cdot x)}$.

(20) If $x - h$, $x \in \operatorname{dom} \operatorname{cosec} \cap \operatorname{dom} \sec$, then $(\nabla_h[\operatorname{cosec} \ \sec])(x) = -4 \cdot \frac{\cos(2 \cdot x-h) \cdot \sin h}{\sin(2 \cdot x) \cdot \sin(2 \cdot (x-h))}$.

(21) If $x + \frac{h}{2}$, $x - \frac{h}{2} \in \operatorname{dom} \operatorname{cosec} \cap \operatorname{dom} \sec$, then $(\delta_h[\operatorname{cosec} \ \sec])(x) = -4 \cdot \frac{\cos(2 \cdot x) \cdot \sin h}{\sin(2 \cdot x+h) \cdot \sin(2 \cdot x-h)}$.

(22) Suppose $x_0 \in$ dom (the function tan) and $x_1 \in$ dom (the function tan). Then Δ[(the function tan) (the function tan) (the function cos)]$(x_0, x_1) =$ Δ[(the function tan) (the function sin)](x_0, x_1).

(23) Suppose $x \in$ dom (the function tan) and $x + h \in$ dom (the function tan). Then $(\Delta_h$[(the function tan) (the function tan) (the function cos)]$)(x) =$

((the function tan) (the function sin))$(x+h)$ − ((the function tan) (the function sin))(x).

(24) Suppose $x \in$ dom (the function tan) and $x - h \in$ dom (the function tan). Then $(\nabla_h$[(the function tan) (the function tan) (the function cos)]$)(x)$ = ((the function tan) (the function sin))(x) − ((the function tan) (the function sin))$(x-h)$.

(25) Suppose $x + \frac{h}{2} \in$ dom (the function tan) and $x - \frac{h}{2} \in$ dom (the function tan). Then $(\delta_h$[(the function tan) (the function tan) (the function cos)]$)(x)$ = ((the function tan) (the function sin))$(x+\frac{h}{2})$ − ((the function tan) (the function sin))$(x-\frac{h}{2})$.

(26) Suppose $x_0 \in$ dom (the function cot) and $x_1 \in$ dom (the function cot). Then Δ[(the function cot) (the function cot) (the function sin)](x_0, x_1) = Δ[(the function cot) (the function cos)](x_0, x_1).

(27) Suppose $x \in$ dom (the function cot) and $x + h \in$ dom (the function cot). Then $(\Delta_h$[(the function cot) (the function cot) (the function sin)]$)(x)$ = ((the function cot) (the function cos))$(x+h)$ − ((the function cot) (the function cos))(x).

(28) Suppose $x \in$ dom (the function cot) and $x - h \in$ dom (the function cot). Then $(\nabla_h$[(the function cot) (the function cot) (the function sin)]$)(x)$ = ((the function cot) (the function cos))(x) − ((the function cot) (the function cos))$(x-h)$.

(29) Suppose $x + \frac{h}{2} \in$ dom (the function cot) and $x - \frac{h}{2} \in$ dom (the function cot). Then $(\delta_h$[(the function cot) (the function cot) (the function sin)]$)(x)$ = ((the function cot) (the function cos))$(x+\frac{h}{2})$ − ((the function cot) (the function cos))$(x-\frac{h}{2})$.

(30) If $x_0 > 0$ and $x_1 > 0$, then Δ[the function ln]$(x_0, x_1) = \frac{(\text{the function ln})(\frac{x_0}{x_1})}{x_0 - x_1}$.

(31) If $x > 0$ and $x + h > 0$, then $(\Delta_h$[the function ln]$)(x)$ = (the function ln)$(1+\frac{h}{x})$.

(32) If $x > 0$ and $x - h > 0$, then $(\nabla_h$[the function ln]$)(x)$ = (the function ln)$(1+\frac{h}{x-h})$.

(33) If $x + \frac{h}{2} > 0$ and $x - \frac{h}{2} > 0$, then $(\delta_h$[the function ln]$)(x)$ = (the function ln)$(1+\frac{h}{x-\frac{h}{2}})$.

(34) For all real numbers h, k holds $\exp(h-k) = \frac{\exp h}{\exp k}$.

(35) $(\Delta_h[f])(x) = (\mathrm{Shift}(f,h))(x) - f(x)$.

(36) If for every x holds $f(x) = (\Delta_h[g])(x)$, then $\Delta[f](x_0, x_1) = \Delta[g](x_0 + h, x_1 + h) - \Delta[g](x_0, x_1)$.

(37) $(\Delta_h[\Delta_h[f]])(x) = (\Delta_{2 \cdot h}[f])(x) - 2 \cdot (\Delta_h[f])(x)$.

(38) $(\nabla_h[\Delta_h[f]])(x) = (\Delta_h[f])(x) - (\nabla_h[f])(x)$.

(39) $(\delta_h[\Delta_h[f]])(x) = (\Delta_h[f])(x + \frac{h}{2}) - (\delta_h[f])(x)$.

(40) $(\vec{\Delta}_h[f])(1)(x) = (\vec{\Delta}_h[f])(0)(x + h) - (\vec{\Delta}_h[f])(0)(x)$.

(41) $(\vec{\Delta}_h[f])(n+1)(x) = (\vec{\Delta}_h[f])(n)(x + h) - (\vec{\Delta}_h[f])(n)(x)$.

(42) $(\nabla_h[f])(x) = f(x) - (\mathrm{Shift}(f, -h))(x)$.

(43) If for every x holds $f(x) = (\nabla_h[g])(x)$, then $\Delta[f](x_0, x_1) = \Delta[g](x_0, x_1) - \Delta[g](x_0 - h, x_1 - h)$.

(44) $(\Delta_h[\nabla_h[f]])(x) = (\Delta_h[f])(x) - (\nabla_h[f])(x)$.

(45) $(\nabla_h[\nabla_h[f]])(x) = 2 \cdot (\nabla_h[f])(x) - (\nabla_{2 \cdot h}[f])(x)$.

(46) $(\delta_h[\nabla_h[f]])(x) = (\delta_h[f])(x) - (\nabla_h[f])(x - \frac{h}{2})$.

(47) $(\vec{\nabla}_h[f])(1)(x) = (\vec{\nabla}_h[f])(0)(x) - (\vec{\nabla}_h[f])(0)(x - h)$.

(48) $(\vec{\nabla}_h[f])(n+1)(x) = (\vec{\nabla}_h[f])(n)(x) - (\vec{\nabla}_h[f])(n)(x - h)$.

(49) $(\delta_h[f])(x) = (\mathrm{Shift}(f, \frac{h}{2}))(x) - (\mathrm{Shift}(f, -\frac{h}{2}))(x)$.

(50) If for every x holds $f(x) = (\delta_h[g])(x)$, then $\Delta[f](x_0, x_1) = \Delta[g](x_0 + \frac{h}{2}, x_1 + \frac{h}{2}) - \Delta[g](x_0 - \frac{h}{2}, x_1 - \frac{h}{2})$.

(51) $(\Delta_h[\delta_h[f]])(x) = (\Delta_h[f])(x + \frac{h}{2}) - (\delta_h[f])(x)$.

(52) $(\nabla_h[\delta_h[f]])(x) = (\delta_h[f])(x) - (\nabla_h[f])(x - \frac{h}{2})$.

(53) $(\delta_h[\delta_h[f]])(x) = (\Delta_h[f])(x) - (\nabla_h[f])(x)$.

(54) $(\vec{\delta}_h[f])(1)(x) = (\vec{\delta}_h[f])(0)(x + \frac{h}{2}) - (\vec{\delta}_h[f])(0)(x - \frac{h}{2})$.

(55) $(\vec{\delta}_h[f])(n+1)(x) = (\vec{\delta}_h[f])(n)(x + \frac{h}{2}) - (\vec{\delta}_h[f])(n)(x - \frac{h}{2})$.

(56) Suppose $x_0 \in \mathrm{dom}$ (the function tan) and $x_1 \in \mathrm{dom}$ (the function tan). Then Δ[(the function tan) (the function tan) (the function sin)]$(x_0, x_1) = \frac{(\sin x_0)^3 \cdot (\cos x_1)^{\mathbf{2}} - (\sin x_1)^3 \cdot (\cos x_0)^{\mathbf{2}}}{(\cos x_0)^{\mathbf{2}} \cdot (\cos x_1)^{\mathbf{2}} \cdot (x_0 - x_1)}$.

(57) Suppose $x \in \mathrm{dom}$ (the function tan) and $x + h \in \mathrm{dom}$ (the function tan). Then $(\Delta_h$[(the function tan) (the function tan) (the function sin)]$)(x)$ = (the function sin)$(x+h)^3 \cdot$ ((the function cos)$(x+h)^{-1})^2$ − (the function sin)$(x)^3 \cdot$ ((the function cos)$(x)^{-1})^2$.

(58) Suppose $x \in \mathrm{dom}$ (the function tan) and $x - h \in \mathrm{dom}$ (the function tan). Then $(\nabla_h$[(the function tan) (the function tan) (the function sin)]$)(x)$ = (the function sin)$(x)^3 \cdot$ ((the function cos)$(x)^{-1})^2$ − (the function sin)$(x-h)^3 \cdot$ ((the function cos)$(x-h)^{-1})^2$.

(59) Suppose $x + \frac{h}{2} \in \mathrm{dom}$ (the function tan) and $x - \frac{h}{2} \in \mathrm{dom}$ (the function tan). Then $(\delta_h$[(the function tan) (the function tan) (the function sin)]$)(x)$ = (the function sin)$(x + \frac{h}{2})^3 \cdot$ ((the function cos)$(x + \frac{h}{2})^{-1})^2$ − (the function sin)$(x - \frac{h}{2})^3 \cdot$ ((the function cos)$(x - \frac{h}{2})^{-1})^2$.

(60) Suppose $x_0 \in \mathrm{dom}$ (the function cot) and $x_1 \in \mathrm{dom}$ (the function cot). Then Δ[(the function cot) (the function cot) (the function cos)]$(x_0, x_1) = \frac{(\cos x_0)^3 \cdot (\sin x_1)^{\mathbf{2}} - (\cos x_1)^3 \cdot (\sin x_0)^{\mathbf{2}}}{(\sin x_0)^{\mathbf{2}} \cdot (\sin x_1)^{\mathbf{2}} \cdot (x_0 - x_1)}$.

(61) Suppose $x \in \text{dom}\,(\text{the function cot})$ and $x + h \in \text{dom}\,(\text{the function cot})$. Then $(\Delta_h[(\text{the function cot})\ (\text{the function cot})\ (\text{the function cos})])(x) = (\text{the function cos})(x+h)^3 \cdot ((\text{the function sin})(x+h)^{-1})^2 - (\text{the function cos})(x)^3 \cdot ((\text{the function sin})(x)^{-1})^2$.

(62) Suppose $x \in \text{dom}\,(\text{the function cot})$ and $x - h \in \text{dom}\,(\text{the function cot})$. Then $(\nabla_h[(\text{the function cot})\ (\text{the function cot})\ (\text{the function cos})])(x) = (\text{the function cos})(x)^3 \cdot ((\text{the function sin})(x)^{-1})^2 - (\text{the function cos})(x-h)^3 \cdot ((\text{the function sin})(x-h)^{-1})^2$.

(63) Suppose $x + \frac{h}{2} \in \text{dom}\,(\text{the function cot})$ and $x - \frac{h}{2} \in \text{dom}\,(\text{the function cot})$. Then $(\delta_h[(\text{the function cot})\ (\text{the function cot})\ (\text{the function cos})])(x) = (\text{the function cos})(x+\frac{h}{2})^3 \cdot ((\text{the function sin})(x+\frac{h}{2})^{-1})^2 - (\text{the function cos})(x-\frac{h}{2})^3 \cdot ((\text{the function sin})(x-\frac{h}{2})^{-1})^2$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.

[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[4] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[5] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[6] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.

[7] Bo Li, Yan Zhang, and Xiquan Liang. Difference and difference quotient. *Formalized Mathematics*, 14(**3**):115–119, 2006, doi:10.2478/v10037-006-0014-z.

[8] Beata Perkowska. Functional sequence from a domain to a domain. *Formalized Mathematics*, 3(**1**):17–21, 1992.

[9] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(**2**):213–216, 1991.

[10] Yasunari Shidama. The Taylor expansions. *Formalized Mathematics*, 12(**2**):195–200, 2004.

[11] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[12] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[13] Peng Wang and Bo Li. Several differentiation formulas of special functions. Part V. *Formalized Mathematics*, 15(**3**):73–79, 2007, doi:10.2478/v10037-007-0009-4.

[14] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

[15] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

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