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Differentiation of Vector-Valued Functions on n-Dimensional Real Normed Linear Spaces

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Summary. In this article, we define and develop differentiation of vector-valued functions on n-dimensional real normed linear spaces (refer to [16] and [17]).

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The papers [8], [14], [2], [3], [4], [5], [13], [18], [1], [12], [6], [10], [15], [11], [9], [21], [19], [20], and [7] provide the terminology and notation for this paper.

1. The Basic Properties of Differentiation of Functions from $\mathcal{R}^m$ to $\mathcal{R}^n$

In this paper i, n, m are elements of $\mathbb{N}$.

The following propositions are true:

(1) Let f be a set. Then f is a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$ if and only if f is a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

(2) Let n, m be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x be an element of $\mathcal{R}^m$, and y be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $f = g$ and $x = y$. Then f is differentiable in x if and only if g is differentiable in y.

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(3) Let n, m be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x be an element of $\mathcal{R}^m$, and y be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. If $f = g$ and $x = y$ and f is differentiable in x, then $f'(x) = g'(y)$.

(4) Let f_1, f_2 be partial functions from $\mathcal{R}^m$ to $\mathcal{R}^n$ and g_1, g_2 be partial functions from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 + f_2 = g_1 + g_2$.

(5) Let f_1, f_2 be partial functions from $\mathcal{R}^m$ to $\mathcal{R}^n$ and g_1, g_2 be partial functions from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 - f_2 = g_1 - g_2$.

(6) Let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and a be a real number. If $f = g$, then $a\,f = a\,g$.

(7) Let f_1, f_2 be functions from $\mathcal{R}^m$ into $\mathcal{R}^n$ and g_1, g_2 be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 + f_2 = g_1 + g_2$.

(8) Let f_1, f_2 be functions from $\mathcal{R}^m$ into $\mathcal{R}^n$ and g_1, g_2 be points of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f_1 = g_1$ and $f_2 = g_2$, then $f_1 - f_2 = g_1 - g_2$.

(9) Let f be a function from $\mathcal{R}^m$ into $\mathcal{R}^n$, g be a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and r be a real number. If $f = g$, then $r\,f = r \cdot g$.

(10) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and x be an element of $\mathcal{R}^m$. Suppose f is differentiable in x. Then $f'(x)$ is a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

Let n, m be natural numbers and let I_1 be a function from $\mathcal{R}^m$ into $\mathcal{R}^n$. We say that I_1 is additive if and only if:

(Def. 1) For all elements x, y of $\mathcal{R}^m$ holds $I_1(x + y) = I_1(x) + I_1(y)$.

We say that I_1 is homogeneous if and only if:

(Def. 2) For every element x of $\mathcal{R}^m$ and for every real number r holds $I_1(r \cdot x) = r \cdot I_1(x)$.

The following three propositions are true:

(11) For every function I_1 from $\mathcal{R}^m$ into $\mathcal{R}^n$ such that I_1 is additive holds $I_1(\underbrace{\langle 0, \ldots, 0 \rangle}_{m}) = \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$.

(12) Let f be a function from $\mathcal{R}^m$ into $\mathcal{R}^n$ and g be a function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f = g$, then f is additive iff g is additive.

(13) Let f be a function from $\mathcal{R}^m$ into $\mathcal{R}^n$ and g be a function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $f = g$, then f is homogeneous iff g is homogeneous.

Let n, m be natural numbers. One can verify that the function $\mathcal{R}^m \longmapsto \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ is additive and homogeneous.

Let n, m be natural numbers. Note that there exists a function from $\mathcal{R}^m$ into $\mathcal{R}^n$ which is additive and homogeneous.

Let m, n be natural numbers. A linear operator from m into n is defined by an additive homogeneous function from $\mathcal{R}^m$ into $\mathcal{R}^n$.

We now state the proposition

(14) Let f be a set. Then f is a linear operator from m into n if and only if f is a linear operator from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$.

Let m, n be natural numbers, let I_1 be a function from $\mathcal{R}^m$ into $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. Then $I_1(x)$ is an element of $\mathcal{R}^n$.

Let m, n be natural numbers and let I_1 be a function from $\mathcal{R}^m$ into $\mathcal{R}^n$. We say that I_1 is bounded if and only if:

(Def. 3) There exists a real number K such that $0 \leq K$ and for every element x of $\mathcal{R}^m$ holds $|I_1(x)| \leq K \cdot |x|$.

Next we state three propositions:

(15) Let x_1, y_1 be finite sequences of elements of $\mathcal{R}^m$. Suppose $\operatorname{len} x_1 = \operatorname{len} y_1 + 1$ and $x_1 \restriction \operatorname{len} y_1 = y_1$. Then there exists an element v of $\mathcal{R}^m$ such that $v = x_1(\operatorname{len} x_1)$ and $\sum x_1 = \sum y_1 + v$.

(16) Let f be a linear operator from m into n, x_1 be a finite sequence of elements of $\mathcal{R}^m$, and y_1 be a finite sequence of elements of $\mathcal{R}^n$. Suppose $\operatorname{len} x_1 = \operatorname{len} y_1$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} x_1$ holds $y_1(i) = f(x_1(i))$. Then $\sum y_1 = f(\sum x_1)$.

(17) Let x_1 be a finite sequence of elements of $\mathcal{R}^m$ and y_1 be a finite sequence of elements of $\mathbb{R}$. Suppose $\operatorname{len} x_1 = \operatorname{len} y_1$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} x_1$ there exists an element v of $\mathcal{R}^m$ such that $v = x_1(i)$ and $y_1(i) = |v|$. Then $|\sum x_1| \leq \sum y_1$.

Let m, n be natural numbers. Note that every linear operator from m into n is bounded.

Let us consider m, n. Observe that every linear operator from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$ is bounded.

Next we state several propositions:

(18) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and x be an element of $\mathcal{R}^m$. Suppose f is differentiable in x. Then $f'(x)$ is a linear operator from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$.

(19) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and x be an element of $\mathcal{R}^m$. Suppose f is differentiable in x. Then $f'(x)$ is a linear operator from m into n.

(20) Let n, m be non empty elements of $\mathbb{N}$, g_1, g_2 be partial functions from

$\mathcal{R}^m$ to $\mathcal{R}^n$, and y_0 be an element of $\mathcal{R}^m$. Suppose g_1 is differentiable in y_0 and g_2 is differentiable in y_0. Then $g_1 + g_2$ is differentiable in y_0 and $(g_1 + g_2)'(y_0) = {g_1}'(y_0) + {g_2}'(y_0)$.

(21) Let n, m be non empty elements of $\mathbb{N}$, g_1, g_2 be partial functions from $\mathcal{R}^m$ to $\mathcal{R}^n$, and y_0 be an element of $\mathcal{R}^m$. Suppose g_1 is differentiable in y_0 and g_2 is differentiable in y_0. Then $g_1 - g_2$ is differentiable in y_0 and $(g_1 - g_2)'(y_0) = {g_1}'(y_0) - {g_2}'(y_0)$.

(22) Let n, m be non empty elements of $\mathbb{N}$, g be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, y_0 be an element of $\mathcal{R}^m$, and r be a real number. Suppose g is differentiable in y_0. Then $r\,g$ is differentiable in y_0 and $(r\,g)'(y_0) = r\,g'(y_0)$.

(23) Let x_0 be an element of $\mathcal{R}^m$, y_0 be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and r be a real number. Suppose $x_0 = y_0$. Then $\{y \in \mathcal{R}^m\colon |y - x_0| < r\} = \{z; z$ ranges over points of $\langle \mathcal{E}^m, \|\cdot\| \rangle\colon \|z - y_0\| < r\}$.

(24) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x_0 be an element of $\mathcal{R}^m$, and L, R be functions from $\mathcal{R}^m$ into $\mathcal{R}^n$. Suppose that

(i) L is a linear operator from m into n, and

(ii) there exists a real number r_0 such that $0 < r_0$ and $\{y \in \mathcal{R}^m\colon |y - x_0| < r_0\} \subseteq \operatorname{dom} f$ and for every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every element z of $\mathcal{R}^m$ and for every element w of $\mathcal{R}^n$ such that $z \neq \langle \underbrace{0, \ldots, 0}_{m} \rangle$ and $|z| < d$ and $w = R(z)$ holds $|z|^{-1} \cdot |w| < r$ and for every element x of $\mathcal{R}^m$ such that $|x - x_0| < r_0$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$.

Then f is differentiable in x_0 and $f'(x_0) = L$.

(25) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and x_0 be an element of $\mathcal{R}^m$. Then f is differentiable in x_0 if and only if there exists a real number r_0 such that $0 < r_0$ and $\{y \in \mathcal{R}^m\colon |y - x_0| < r_0\} \subseteq \operatorname{dom} f$ and there exist functions L, R from $\mathcal{R}^m$ into $\mathcal{R}^n$ such that L is a linear operator from m into n and for every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every element z of $\mathcal{R}^m$ and for every element w of $\mathcal{R}^n$ such that $z \neq \langle \underbrace{0, \ldots, 0}_{m} \rangle$ and $|z| < d$ and $w = R(z)$ holds $|z|^{-1} \cdot |w| < r$ and for every element x of $\mathcal{R}^m$ such that $|x - x_0| < r_0$ holds $f(x) - f(x_0) = L(x - x_0) + R(x - x_0)$.

2. Differentiation of Functions from Normed Linear Spaces $\mathcal{R}^m$ to Normed Linear Spaces $\mathcal{R}^n$

One can prove the following propositions:

(26) For all points y_2, y_3 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ holds $(\mathrm{Proj}(i,n))(y_2 + y_3) = (\mathrm{Proj}(i,n))(y_2) + (\mathrm{Proj}(i,n))(y_3)$.

(27) For every point y_2 of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and for every real number r holds $(\mathrm{Proj}(i,n))(r \cdot y_2) = r \cdot (\mathrm{Proj}(i,n))(y_2)$.

(28) Let m, n be non empty elements of $\mathbb{N}$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x_0 be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and i be an element of $\mathbb{N}$. Suppose $1 \leq i \leq n$ and g is differentiable in x_0. Then $\mathrm{Proj}(i,n) \cdot g$ is differentiable in x_0 and $\mathrm{Proj}(i,n) \cdot g'(x_0) = (\mathrm{Proj}(i,n) \cdot g)'(x_0)$.

(29) Let m, n be non empty elements of $\mathbb{N}$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and x_0 be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Then g is differentiable in x_0 if and only if for every element i of $\mathbb{N}$ such that $1 \leq i \leq n$ holds $\mathrm{Proj}(i,n) \cdot g$ is differentiable in x_0.

Let X be a set, let n, m be non empty elements of $\mathbb{N}$, and let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$. We say that f is differentiable on X if and only if:

(Def. 4) $X \subseteq \mathrm{dom}\, f$ and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $f{\restriction}X$ is differentiable in x.

The following four propositions are true:

(30) Let X be a set, m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $f = g$. Then f is differentiable on X if and only if g is differentiable on X.

(31) Let X be a set, m, n be non empty elements of $\mathbb{N}$, and f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$. If f is differentiable on X, then X is a subset of $\mathcal{R}^m$.

(32) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and Z be a subset of $\mathcal{R}^m$. Given a subset Z_0 of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $Z = Z_0$ and Z_0 is open. Then f is differentiable on Z if and only if the following conditions are satisfied:

(i) $Z \subseteq \mathrm{dom}\, f$, and

(ii) for every element x of $\mathcal{R}^m$ such that $x \in Z$ holds f is differentiable in x.

(33) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and Z be a subset of $\mathcal{R}^m$. Suppose f is differentiable on Z. Then there exists a subset Z_0 of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $Z = Z_0$ and Z_0 is open.

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