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Nilpotent Groups

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Summary. This article describes the concept of the nilpotent group and some properties of the nilpotent groups.

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The papers [2], [3], [4], [6], [7], [5], [8], [9], [10], and [1] provide the terminology and notation for this paper.

For simplicity, we adopt the following convention: x denotes a set, G denotes a group, A, B, H, H_1, H_2 denote subgroups of G, a, b, c denote elements of G, F denotes a finite sequence of elements of the carrier of G, and i, j denote elements of $\mathbb{N}$.

One can prove the following propositions:

(1) $a^b = a \cdot [a, b]$.

(2) $[a, b]^{-1} = [a, b^{-1}]^b$.

(3) $[a, b]^{-1} = [a^{-1}, b]^a$.

(4) $([a, b^{-1}]^b)^{-1} = [b^{-1}, a]^b$.

(5) $[a,\ b^{-1},\ c]^b = [[a, b^{-1}]^b, c^b]$.

(6) $[a, b^{-1}]^b = [b, a]$.

(7) $[a,\ b^{-1},\ c]^b = [b,\ a,\ c^b]$.

(8) $[a,\ b,\ c^a] \cdot [c,\ a,\ b^c] \cdot [b,\ c,\ a^b] = \mathbf{1}_G$.

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(9) $[A, B]$ is a subgroup of $[B, A]$.

(10) $[A, B] = [B, A]$.

Let us consider G, A, B. Let us note that the functor $[A, B]$ is commutative.

One can prove the following propositions:

(11) If B is a subgroup of A, then the commutators of A & $B \subseteq \overline{A}$.

(12) If B is a subgroup of A, then $[A, B]$ is a subgroup of A.

(13) If B is a subgroup of A, then $[B, A]$ is a subgroup of A.

(14) If $[H_1, \Omega_G]$ is a subgroup of H_2, then $[H_1 \cap H, H]$ is a subgroup of $H_2 \cap H$.

(15) $[H_1, H_2]$ is a subgroup of $[H_1, \Omega_G]$.

(16) A is a normal subgroup of G iff $[A, \Omega_G]$ is a subgroup of A.

Let us consider G. The normal subgroups of G yields a set and is defined by:

(Def. 1) $x \in$ the normal subgroups of G iff x is a strict normal subgroup of G.

Let us consider G. One can verify that the normal subgroups of G is non empty.

Next we state three propositions:

(17) Let F be a finite sequence of elements of the normal subgroups of G and given j. If $j \in \operatorname{dom} F$, then $F(j)$ is a strict normal subgroup of G.

(18) The normal subgroups of $G \subseteq \operatorname{SubGr} G$.

(19) Every finite sequence of elements of the normal subgroups of G is a finite sequence of elements of $\operatorname{SubGr} G$.

Let I_1 be a group. We say that I_1 is nilpotent if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exists a finite sequence F of elements of the normal subgroups of I_1 such that

(i) $\operatorname{len} F > 0$,

(ii) $F(1) = \Omega_{(I_1)}$,

(iii) $F(\operatorname{len} F) = \{\mathbf{1}\}_{(I_1)}$, and

(iv) for every i such that i, $i+1 \in \operatorname{dom} F$ and for all strict normal subgroups G_1, G_2 of I_1 such that $G_1 = F(i)$ and $G_2 = F(i+1)$ holds G_2 is a subgroup of G_1 and ${}^{G_1}/_{(G_2)_{(G_1)}}$ is a subgroup of $\mathrm{Z}({}^{I_1}/_{G_2})$.

Let us note that there exists a group which is nilpotent and strict.

We now state four propositions:

(20) Let G_1 be a subgroup of G and N be a strict normal subgroup of G. Suppose N is a subgroup of G_1 and ${}^{G_1}/_{(N)_{(G_1)}}$ is a subgroup of $\mathrm{Z}({}^{G}/_{N})$. Then $[G_1, \Omega_G]$ is a subgroup of N.

(21) Let G_1 be a subgroup of G and N be a normal subgroup of G. Suppose N is a strict subgroup of G_1 and $[G_1, \Omega_G]$ is a strict subgroup of N. Then ${}^{G_1}/_{(N)_{(G_1)}}$ is a subgroup of $\mathrm{Z}({}^{G}/_{N})$.

(22) Let G be a group. Then G is nilpotent if and only if there exists a finite sequence F of elements of the normal subgroups of G such that $\operatorname{len} F > 0$ and $F(1) = \Omega_G$ and $F(\operatorname{len} F) = \{\mathbf{1}\}_G$ and for every i such that i, $i+1 \in \operatorname{dom} F$ and for all strict normal subgroups G_1, G_2 of G such that $G_1 = F(i)$ and $G_2 = F(i+1)$ holds G_2 is a subgroup of G_1 and $[G_1, \Omega_G]$ is a subgroup of G_2.

(23) Let G be a group, H, G_1 be subgroups of G, G_2 be a strict normal subgroup of G, H_1 be a subgroup of H, and H_2 be a normal subgroup of H. Suppose G_2 is a subgroup of G_1 and ${}^{G_1}/_{(G_2)_{(G_1)}}$ is a subgroup of $\mathrm{Z}({}^{G}/_{G_2})$ and $H_1 = G_1 \cap H$ and $H_2 = G_2 \cap H$. Then ${}^{H_1}/_{(H_2)_{(H_1)}}$ is a subgroup of $\mathrm{Z}({}^{H}/_{H_2})$.

Let G be a nilpotent group. Note that every subgroup of G is nilpotent.

Let us mention that every group which is commutative is also nilpotent and every group which is cyclic is also nilpotent.

We now state four propositions:

(24) Let G, H be strict groups, h be a homomorphism from G to H, A be a strict subgroup of G, and a, b be elements of G. Then $h(a) \cdot h(b) \cdot h^{\circ}A = h^{\circ}(a \cdot b \cdot A)$ and $h^{\circ}A \cdot h(a) \cdot h(b) = h^{\circ}(A \cdot a \cdot b)$.

(25) Let G, H be strict groups, h be a homomorphism from G to H, A be a strict subgroup of G, a, b be elements of G, H_1 be a subgroup of $\operatorname{Im} h$, and a_1, b_1 be elements of $\operatorname{Im} h$. If $a_1 = h(a)$ and $b_1 = h(b)$ and $H_1 = h^{\circ}A$, then $a_1 \cdot b_1 \cdot H_1 = h(a) \cdot h(b) \cdot h^{\circ}A$.

(26) Let G, H be strict groups, h be a homomorphism from G to H, G_1 be a strict subgroup of G, G_2 be a strict normal subgroup of G, H_1 be a strict subgroup of $\operatorname{Im} h$, and H_2 be a strict normal subgroup of $\operatorname{Im} h$. Suppose G_2 is a strict subgroup of G_1 and ${}^{G_1}/_{(G_2)_{(G_1)}}$ is a subgroup of $\mathrm{Z}({}^{G}/_{G_2})$ and $H_1 = h^{\circ}G_1$ and $H_2 = h^{\circ}G_2$. Then ${}^{H_1}/_{(H_2)_{(H_1)}}$ is a subgroup of $\mathrm{Z}({}^{\operatorname{Im} h}/_{H_2})$.

(27) Let G, H be strict groups, h be a homomorphism from G to H, and A be a strict normal subgroup of G. Then $h^{\circ}A$ is a strict normal subgroup of $\operatorname{Im} h$.

Let G be a strict nilpotent group, let H be a strict group, and let h be a homomorphism from G to H. One can check that $\operatorname{Im} h$ is nilpotent.

Let G be a strict nilpotent group and let N be a strict normal subgroup of G. Note that ${}^{G}/_{N}$ is nilpotent.

One can prove the following three propositions:

(28) Let G be a group. Given a finite sequence F of elements of the normal subgroups of G such that

(i) $\operatorname{len} F > 0$,

(ii) $F(1) = \Omega_G$,

(iii) $F(\operatorname{len} F) = \{\mathbf{1}\}_G$, and

(iv) for every i such that i, $i+1 \in \operatorname{dom} F$ and for every strict normal subgroup G_1 of G such that $G_1 = F(i)$ holds $[G_1, \Omega_G] = F(i+1)$.
Then G is nilpotent.

(29) Let G be a group. Given a finite sequence F of elements of the normal subgroups of G such that
(i) $\operatorname{len} F > 0$,
(ii) $F(1) = \Omega_G$,
(iii) $F(\operatorname{len} F) = \{\mathbf{1}\}_G$, and
(iv) for every i such that i, $i+1 \in \operatorname{dom} F$ and for all strict normal subgroups G_1, G_2 of G such that $G_1 = F(i)$ and $G_2 = F(i+1)$ holds G_2 is a subgroup of G_1 and ${}^{G}/_{G_2}$ is a commutative group.
Then G is nilpotent.

(30) Let G be a group. Given a finite sequence F of elements of the normal subgroups of G such that
(i) $\operatorname{len} F > 0$,
(ii) $F(1) = \Omega_G$,
(iii) $F(\operatorname{len} F) = \{\mathbf{1}\}_G$, and
(iv) for every i such that i, $i+1 \in \operatorname{dom} F$ and for all strict normal subgroups G_1, G_2 of G such that $G_1 = F(i)$ and $G_2 = F(i+1)$ holds G_2 is a subgroup of G_1 and ${}^{G}/_{G_2}$ is a cyclic group.
Then G is nilpotent.

Let us mention that every group which is nilpotent is also solvable.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[5] Wojciech A. Trybulec. Classes of conjugation. Normal subgroups. *Formalized Mathematics*, 1(**5**):955–962, 1990.
[6] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[7] Wojciech A. Trybulec. Subgroup and cosets of subgroups. *Formalized Mathematics*, 1(**5**):855–864, 1990.
[8] Wojciech A. Trybulec. Commutator and center of a group. *Formalized Mathematics*, 2(**4**):461–466, 1991.
[9] Wojciech A. Trybulec and Michał J. Trybulec. Homomorphisms and isomorphisms of groups. Quotient group. *Formalized Mathematics*, 2(**4**):573–578, 1991.
[10] Katarzyna Zawadzka. Solvable groups. *Formalized Mathematics*, 5(**1**):145–147, 1996.

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