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Second-Order Partial Differentiation of Real Binary Functions

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Summary. In this article we define second-order partial differentiation of real binary functions and discuss the relation of second-order partial derivatives and partial derivatives defined in [17].

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The articles [15], [3], [4], [16], [5], [10], [1], [8], [11], [9], [2], [14], [6], [13], [12], [7], and [17] provide the notation and terminology for this paper.

1. Second-Order Partial Derivatives

For simplicity, we adopt the following convention: x, x_0, y, y_0, r are real numbers, z, z_0 are elements of $\mathcal{R}^2$, f, f_1, f_2 are partial functions from $\mathcal{R}^2$ to $\mathbb{R}$, R is a rest, and L is a linear function.

Let us note that every rest is total.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. The functor $\mathrm{pdiff1}(f, z)$ yielding a function from $\mathcal{R}^2$ into $\mathbb{R}$ is defined as follows:

(Def. 1) For every z such that $z \in \mathcal{R}^2$ holds $(\mathrm{pdiff1}(f, z))(z) = \mathrm{partdiff1}(f, z)$.

The functor $\mathrm{pdiff2}(f, z)$ yields a function from $\mathcal{R}^2$ into $\mathbb{R}$ and is defined as follows:

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(Def. 2) For every z such that $z \in \mathcal{R}^2$ holds $(\mathrm{pdiff2}(f,z))(z) = \mathrm{partdiff2}(f,z)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. We say that f is partial differentiable on 1st-1st coordinate in z if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exist real numbers x_0, y_0 such that
- (i) $z = \langle x_0, y_0 \rangle$, and
- (ii) there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(\mathrm{pdiff1}(f,z),z)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(\mathrm{pdiff1}(f,z),z))(x) - (\mathrm{SVF1}(\mathrm{pdiff1}(f,z),z))(x_0) = L(x - x_0) + R(x - x_0)$.

We say that f is partial differentiable on 1st-2nd coordinate in z if and only if the condition (Def. 4) is satisfied.

(Def. 4) There exist real numbers x_0, y_0 such that
- (i) $z = \langle x_0, y_0 \rangle$, and
- (ii) there exists a neighbourhood N of y_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF2}(\mathrm{pdiff1}(f,z),z)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF2}(\mathrm{pdiff1}(f,z),z))(y) - (\mathrm{SVF2}(\mathrm{pdiff1}(f,z),z))(y_0) = L(y - y_0) + R(y - y_0)$.

We say that f is partial differentiable on 2nd-1st coordinate in z if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exist real numbers x_0, y_0 such that
- (i) $z = \langle x_0, y_0 \rangle$, and
- (ii) there exists a neighbourhood N of x_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(\mathrm{pdiff2}(f,z),z)$ and there exist L, R such that for every x such that $x \in N$ holds $(\mathrm{SVF1}(\mathrm{pdiff2}(f,z),z))(x) - (\mathrm{SVF1}(\mathrm{pdiff2}(f,z),z))(x_0) = L(x - x_0) + R(x - x_0)$.

We say that f is partial differentiable on 2nd-2nd coordinate in z if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exist real numbers x_0, y_0 such that
- (i) $z = \langle x_0, y_0 \rangle$, and
- (ii) there exists a neighbourhood N of y_0 such that $N \subseteq \mathrm{dom}\,\mathrm{SVF2}(\mathrm{pdiff2}(f,z),z)$ and there exist L, R such that for every y such that $y \in N$ holds $(\mathrm{SVF2}(\mathrm{pdiff2}(f,z),z))(y) - (\mathrm{SVF2}(\mathrm{pdiff2}(f,z),z))(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. Let us assume that f is partial differentiable on 1st-1st coordinate in z. The functor $\mathrm{hpartdiff11}(f,z)$ yields a real number and is defined by the condition (Def. 7).

(Def. 7) There exist real numbers x_0, y_0 such that
- (i) $z = \langle x_0, y_0 \rangle$, and

(ii) there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(\mathrm{pdiff1}(f, z), z)$ and there exist L, R such that $\mathrm{hpartdiff11}(f, z) = L(1)$ and for every x such that $x \in N$ holds $(\mathrm{SVF1}(\mathrm{pdiff1}(f, z), z))(x) - (\mathrm{SVF1}(\mathrm{pdiff1}(f, z), z))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. Let us assume that f is partial differentiable on 1st-2nd coordinate in z. The functor $\mathrm{hpartdiff12}(f, z)$ yielding a real number is defined by the condition (Def. 8).

(Def. 8) There exist real numbers x_0, y_0 such that

(i) $z = \langle x_0, y_0 \rangle$, and

(ii) there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF2}(\mathrm{pdiff1}(f, z), z)$ and there exist L, R such that $\mathrm{hpartdiff12}(f, z) = L(1)$ and for every y such that $y \in N$ holds $(\mathrm{SVF2}(\mathrm{pdiff1}(f, z), z))(y) - (\mathrm{SVF2}(\mathrm{pdiff1}(f, z), z))(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. Let us assume that f is partial differentiable on 2nd-1st coordinate in z. The functor $\mathrm{hpartdiff21}(f, z)$ yielding a real number is defined by the condition (Def. 9).

(Def. 9) There exist real numbers x_0, y_0 such that

(i) $z = \langle x_0, y_0 \rangle$, and

(ii) there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF1}(\mathrm{pdiff2}(f, z), z)$ and there exist L, R such that $\mathrm{hpartdiff21}(f, z) = L(1)$ and for every x such that $x \in N$ holds $(\mathrm{SVF1}(\mathrm{pdiff2}(f, z), z))(x) - (\mathrm{SVF1}(\mathrm{pdiff2}(f, z), z))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. Let us assume that f is partial differentiable on 2nd-2nd coordinate in z. The functor $\mathrm{hpartdiff22}(f, z)$ yields a real number and is defined by the condition (Def. 10).

(Def. 10) There exist real numbers x_0, y_0 such that

(i) $z = \langle x_0, y_0 \rangle$, and

(ii) there exists a neighbourhood N of y_0 such that $N \subseteq \operatorname{dom} \mathrm{SVF2}(\mathrm{pdiff2}(f, z), z)$ and there exist L, R such that $\mathrm{hpartdiff22}(f, z) = L(1)$ and for every y such that $y \in N$ holds $(\mathrm{SVF2}(\mathrm{pdiff2}(f, z), z))(y) - (\mathrm{SVF2}(\mathrm{pdiff2}(f, z), z))(y_0) = L(y - y_0) + R(y - y_0)$.

Next we state several propositions:

(1) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st-1st coordinate in z, then $\mathrm{SVF1}(\mathrm{pdiff1}(f, z), z)$ is differentiable in x_0.

(2) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st-2nd coordinate in z, then $\mathrm{SVF2}(\mathrm{pdiff1}(f, z), z)$ is differentiable in y_0.

(3) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd-1st coordinate in z, then $\mathrm{SVF1}(\mathrm{pdiff2}(f, z), z)$ is differentiable in x_0.

(4) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd-2nd coordinate in z, then SVF2(pdiff2$(f, z), z)$ is differentiable in y_0.

(5) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st-1st coordinate in z, then hpartdiff11$(f, z) = ($SVF1(pdiff1$(f, z), z))'(x_0)$.

(6) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st-2nd coordinate in z, then hpartdiff12$(f, z) = ($SVF2(pdiff1$(f, z), z))'(y_0)$.

(7) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd-1st coordinate in z, then hpartdiff21$(f, z) = ($SVF1(pdiff2$(f, z), z))'(x_0)$.

(8) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd-2nd coordinate in z, then hpartdiff22$(f, z) = ($SVF2(pdiff2$(f, z), z))'(y_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. We say that f is partial differentiable on 1st-1st coordinate on Z if and only if:

(Def. 11) $Z \subseteq \operatorname{dom} f$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f{\restriction}Z$ is partial differentiable on 1st-1st coordinate in z.

We say that f is partial differentiable on 1st-2nd coordinate on Z if and only if:

(Def. 12) $Z \subseteq \operatorname{dom} f$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f{\restriction}Z$ is partial differentiable on 1st-2nd coordinate in z.

We say that f is partial differentiable on 2nd-1st coordinate on Z if and only if:

(Def. 13) $Z \subseteq \operatorname{dom} f$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f{\restriction}Z$ is partial differentiable on 2nd-1st coordinate in z.

We say that f is partial differentiable on 2nd-2nd coordinate on Z if and only if:

(Def. 14) $Z \subseteq \operatorname{dom} f$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f{\restriction}Z$ is partial differentiable on 2nd-2nd coordinate in z.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. Let us assume that f is partial differentiable on 1st-1st coordinate on Z. The functor $f{}^{\mathrm{1st-1st}}_{\restriction Z}$ yields a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and is defined by:

(Def. 15) $\operatorname{dom}(f{}^{\mathrm{1st-1st}}_{\restriction Z}) = Z$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f{}^{\mathrm{1st-1st}}_{\restriction Z}(z) = \mathrm{hpartdiff11}(f, z)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. Let us assume that f is partial differentiable on 1st-2nd coordinate on Z. The functor $f{}^{\mathrm{1st-2nd}}_{\restriction Z}$ yielding a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ is defined by:

(Def. 16) $\operatorname{dom}(f{}^{\mathrm{1st-2nd}}_{\restriction Z}) = Z$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f{}^{\mathrm{1st-2nd}}_{\restriction Z}(z) = \mathrm{hpartdiff12}(f, z)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. Let us assume that f is partial differentiable on 2nd-1st coordinate on Z. The functor $f{}^{\mathrm{2nd-1st}}_{\restriction Z}$ yields a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and is defined by:

(Def. 17) $\mathrm{dom}(f_{\restriction Z}^{\mathrm{2nd-1st}}) = Z$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f_{\restriction Z}^{\mathrm{2nd-1st}}(z) = \mathrm{hpartdiff21}(f, z)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. Let us assume that f is partial differentiable on 2nd-2nd coordinate on Z. The functor $f_{\restriction Z}^{\mathrm{2nd-2nd}}$ yields a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and is defined by:

(Def. 18) $\mathrm{dom}(f_{\restriction Z}^{\mathrm{2nd-2nd}}) = Z$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f_{\restriction Z}^{\mathrm{2nd-2nd}}(z) = \mathrm{hpartdiff22}(f, z)$.

2. Main Properties of Second-Order Partial Derivatives

One can prove the following propositions:

(9) f is partial differentiable on 1st-1st coordinate in z if and only if $\mathrm{pdiff1}(f, z)$ is partial differentiable on 1st coordinate in z.

(10) f is partial differentiable on 1st-2nd coordinate in z if and only if $\mathrm{pdiff1}(f, z)$ is partial differentiable on 2nd coordinate in z.

(11) f is partial differentiable on 2nd-1st coordinate in z if and only if $\mathrm{pdiff2}(f, z)$ is partial differentiable on 1st coordinate in z.

(12) f is partial differentiable on 2nd-2nd coordinate in z if and only if $\mathrm{pdiff2}(f, z)$ is partial differentiable on 2nd coordinate in z.

(13) f is partial differentiable on 1st-1st coordinate in z if and only if $\mathrm{pdiff1}(f, z)$ is partially differentiable in z w.r.t. coordinate 1.

(14) f is partial differentiable on 1st-2nd coordinate in z if and only if $\mathrm{pdiff1}(f, z)$ is partially differentiable in z w.r.t. coordinate 2.

(15) f is partial differentiable on 2nd-1st coordinate in z if and only if $\mathrm{pdiff2}(f, z)$ is partially differentiable in z w.r.t. coordinate 1.

(16) f is partial differentiable on 2nd-2nd coordinate in z if and only if $\mathrm{pdiff2}(f, z)$ is partially differentiable in z w.r.t. coordinate 2.

(17) If f is partial differentiable on 1st-1st coordinate in z, then $\mathrm{hpartdiff11}(f, z) = \mathrm{partdiff1}(\mathrm{pdiff1}(f, z), z)$.

(18) If f is partial differentiable on 1st-2nd coordinate in z, then $\mathrm{hpartdiff12}(f, z) = \mathrm{partdiff2}(\mathrm{pdiff1}(f, z), z)$.

(19) If f is partial differentiable on 2nd-1st coordinate in z, then $\mathrm{hpartdiff21}(f, z) = \mathrm{partdiff1}(\mathrm{pdiff2}(f, z), z)$.

(20) If f is partial differentiable on 2nd-2nd coordinate in z, then $\mathrm{hpartdiff22}(f, z) = \mathrm{partdiff2}(\mathrm{pdiff2}(f, z), z)$.

(21) Let z_0 be an element of $\mathcal{R}^2$ and N be a neighbourhood of $(\mathrm{proj}(1, 2))(z_0)$. Suppose f is partial differentiable on 1st-1st coordinate in z_0 and $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(\mathrm{pdiff1}(f, z_0), z_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\mathrm{rng}\, c =$

$\{(\text{proj}(1,2))(z_0)\}$ and $\text{rng}(h+c) \subseteq N$. Then h^{-1} (SVF1(pdiff1(f, z_0), z_0) $\cdot$ $(h+c)$ − SVF1(pdiff1(f, z_0), z_0) $\cdot\, c$) is convergent and hpartdiff11(f, z_0) = lim(h^{-1} (SVF1(pdiff1(f, z_0), z_0) $\cdot$ $(h+c)$ − SVF1(pdiff1(f, z_0), z_0) $\cdot\, c$)).

(22) Let z_0 be an element of $\mathcal{R}^2$ and N be a neighbourhood of $(\text{proj}(2,2))(z_0)$. Suppose f is partial differentiable on 1st-2nd coordinate in z_0 and $N \subseteq$ dom SVF2(pdiff1(f, z_0), z_0). Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng}\, c = \{(\text{proj}(2,2))(z_0)\}$ and $\text{rng}(h+c) \subseteq N$. Then h^{-1} (SVF2(pdiff1(f, z_0), z_0) $\cdot$ $(h+c)$ − SVF2(pdiff1(f, z_0), z_0) $\cdot\, c$) is convergent and hpartdiff12(f, z_0) = lim(h^{-1} (SVF2(pdiff1(f, z_0), z_0) $\cdot$ $(h+c)$ − SVF2(pdiff1(f, z_0), z_0) $\cdot\, c$)).

(23) Let z_0 be an element of $\mathcal{R}^2$ and N be a neighbourhood of $(\text{proj}(1,2))(z_0)$. Suppose f is partial differentiable on 2nd-1st coordinate in z_0 and $N \subseteq$ dom SVF1(pdiff2(f, z_0), z_0). Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng}\, c = \{(\text{proj}(1,2))(z_0)\}$ and $\text{rng}(h+c) \subseteq N$. Then h^{-1} (SVF1(pdiff2(f, z_0), z_0) $\cdot$ $(h+c)$ − SVF1(pdiff2(f, z_0), z_0) $\cdot\, c$) is convergent and hpartdiff21(f, z_0) = lim(h^{-1} (SVF1(pdiff2(f, z_0), z_0) $\cdot$ $(h+c)$ − SVF1(pdiff2(f, z_0), z_0) $\cdot\, c$)).

(24) Let z_0 be an element of $\mathcal{R}^2$ and N be a neighbourhood of $(\text{proj}(2,2))(z_0)$. Suppose f is partial differentiable on 2nd-2nd coordinate in z_0 and $N \subseteq$ dom SVF2(pdiff2(f, z_0), z_0). Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng}\, c = \{(\text{proj}(2,2))(z_0)\}$ and $\text{rng}(h+c) \subseteq N$. Then h^{-1} (SVF2(pdiff2(f, z_0), z_0) $\cdot$ $(h+c)$ − SVF2(pdiff2(f, z_0), z_0) $\cdot\, c$) is convergent and hpartdiff22(f, z_0) = lim(h^{-1} (SVF2(pdiff2(f, z_0), z_0) $\cdot$ $(h+c)$ − SVF2(pdiff2(f, z_0), z_0) $\cdot\, c$)).

(25) Suppose that
  (i) f_1 is partial differentiable on 1st-1st coordinate in z_0, and
  (ii) f_2 is partial differentiable on 1st-1st coordinate in z_0.

Then pdiff1(f_1, z_0)+pdiff1(f_2, z_0) is partial differentiable on 1st coordinate in z_0 and partdiff1(pdiff1(f_1, z_0)+pdiff1(f_2, z_0), z_0) = hpartdiff11(f_1, z_0)+ hpartdiff11(f_2, z_0).

(26) Suppose that
  (i) f_1 is partial differentiable on 1st-2nd coordinate in z_0, and
  (ii) f_2 is partial differentiable on 1st-2nd coordinate in z_0.

Then pdiff1(f_1, z_0) + pdiff1(f_2, z_0) is partial differentiable on 2nd coordinate in z_0 and partdiff2(pdiff1(f_1, z_0) + pdiff1(f_2, z_0), z_0) = hpartdiff12(f_1, z_0) + hpartdiff12(f_2, z_0).

(27) Suppose that
  (i) f_1 is partial differentiable on 2nd-1st coordinate in z_0, and
  (ii) f_2 is partial differentiable on 2nd-1st coordinate in z_0.

Then pdiff2(f_1, z_0)+pdiff2(f_2, z_0) is partial differentiable on 1st coordinate in z_0 and partdiff1(pdiff2(f_1, z_0)+pdiff2(f_2, z_0), z_0) = hpartdiff21(f_1, z_0)+

hpartdiff21(f_2, z_0).

(28) Suppose that
(i) f_1 is partial differentiable on 2nd-2nd coordinate in z_0, and
(ii) f_2 is partial differentiable on 2nd-2nd coordinate in z_0.
Then pdiff2(f_1, z_0) + pdiff2(f_2, z_0) is partial differentiable on 2nd coordinate in z_0 and partdiff2(pdiff2(f_1, z_0) + pdiff2$(f_2, z_0), z_0)$ = hpartdiff22(f_1, z_0) + hpartdiff22(f_2, z_0).

(29) Suppose that
(i) f_1 is partial differentiable on 1st-1st coordinate in z_0, and
(ii) f_2 is partial differentiable on 1st-1st coordinate in z_0.
Then pdiff1(f_1, z_0) − pdiff1(f_2, z_0) is partial differentiable on 1st coordinate in z_0 and partdiff1(pdiff1(f_1, z_0) − pdiff1$(f_2, z_0), z_0)$ = hpartdiff11(f_1, z_0) − hpartdiff11(f_2, z_0).

(30) Suppose that
(i) f_1 is partial differentiable on 1st-2nd coordinate in z_0, and
(ii) f_2 is partial differentiable on 1st-2nd coordinate in z_0.
Then pdiff1(f_1, z_0) − pdiff1(f_2, z_0) is partial differentiable on 2nd coordinate in z_0 and partdiff2(pdiff1(f_1, z_0) − pdiff1$(f_2, z_0), z_0)$ = hpartdiff12(f_1, z_0) − hpartdiff12(f_2, z_0).

(31) Suppose that
(i) f_1 is partial differentiable on 2nd-1st coordinate in z_0, and
(ii) f_2 is partial differentiable on 2nd-1st coordinate in z_0.
Then pdiff2(f_1, z_0) − pdiff2(f_2, z_0) is partial differentiable on 1st coordinate in z_0 and partdiff1(pdiff2(f_1, z_0) − pdiff2$(f_2, z_0), z_0)$ = hpartdiff21(f_1, z_0) − hpartdiff21(f_2, z_0).

(32) Suppose that
(i) f_1 is partial differentiable on 2nd-2nd coordinate in z_0, and
(ii) f_2 is partial differentiable on 2nd-2nd coordinate in z_0.
Then pdiff2(f_1, z_0) − pdiff2(f_2, z_0) is partial differentiable on 2nd coordinate in z_0 and partdiff2(pdiff2(f_1, z_0) − pdiff2$(f_2, z_0), z_0)$ = hpartdiff22(f_1, z_0) − hpartdiff22(f_2, z_0).

(33) Suppose f is partial differentiable on 1st-1st coordinate in z_0. Then r pdiff1(f, z_0) is partial differentiable on 1st coordinate in z_0 and partdiff1(r pdiff1$(f, z_0), z_0) = r \cdot$ hpartdiff11(f, z_0).

(34) Suppose f is partial differentiable on 1st-2nd coordinate in z_0. Then r pdiff1(f, z_0) is partial differentiable on 2nd coordinate in z_0 and partdiff2(r pdiff1$(f, z_0), z_0) = r \cdot$ hpartdiff12(f, z_0).

(35) Suppose f is partial differentiable on 2nd-1st coordinate in z_0. Then r pdiff2(f, z_0) is partial differentiable on 1st coordinate in z_0 and partdiff1(r pdiff2$(f, z_0), z_0) = r \cdot$ hpartdiff21(f, z_0).

(36) Suppose f is partial differentiable on 2nd-2nd coordinate in z_0. Then r pdiff2(f, z_0) is partial differentiable on 2nd coordinate in z_0 and partdiff2(r pdiff2$(f, z_0), z_0) = r \cdot$ hpartdiff22(f, z_0).

(37) Suppose that
  (i) f_1 is partial differentiable on 1st-1st coordinate in z_0, and
  (ii) f_2 is partial differentiable on 1st-1st coordinate in z_0.

  Then pdiff1(f_1, z_0) pdiff1(f_2, z_0) is partial differentiable on 1st coordinate in z_0.

(38) Suppose that
  (i) f_1 is partial differentiable on 1st-2nd coordinate in z_0, and
  (ii) f_2 is partial differentiable on 1st-2nd coordinate in z_0.

  Then pdiff1(f_1, z_0) pdiff1(f_2, z_0) is partial differentiable on 2nd coordinate in z_0.

(39) Suppose that
  (i) f_1 is partial differentiable on 2nd-1st coordinate in z_0, and
  (ii) f_2 is partial differentiable on 2nd-1st coordinate in z_0.

  Then pdiff2(f_1, z_0) pdiff2(f_2, z_0) is partial differentiable on 1st coordinate in z_0.

(40) Suppose that
  (i) f_1 is partial differentiable on 2nd-2nd coordinate in z_0, and
  (ii) f_2 is partial differentiable on 2nd-2nd coordinate in z_0.

  Then pdiff2(f_1, z_0) pdiff2(f_2, z_0) is partial differentiable on 2nd coordinate in z_0.

(41) Let z_0 be an element of $\mathcal{R}^2$. Suppose f is partial differentiable on 1st-1st coordinate in z_0. Then SVF1(pdiff1$(f, z_0), z_0)$ is continuous in (proj$(1, 2))(z_0)$.

(42) Let z_0 be an element of $\mathcal{R}^2$. Suppose f is partial differentiable on 1st-2nd coordinate in z_0. Then SVF2(pdiff1$(f, z_0), z_0)$ is continuous in (proj$(2, 2))(z_0)$.

(43) Let z_0 be an element of $\mathcal{R}^2$. Suppose f is partial differentiable on 2nd-1st coordinate in z_0. Then SVF1(pdiff2$(f, z_0), z_0)$ is continuous in (proj$(1, 2))(z_0)$.

(44) Let z_0 be an element of $\mathcal{R}^2$. Suppose f is partial differentiable on 2nd-2nd coordinate in z_0. Then SVF2(pdiff2$(f, z_0), z_0)$ is continuous in (proj$(2, 2))(z_0)$.

(45) If f is partial differentiable on 1st-1st coordinate in z_0, then there exists R such that $R(0) = 0$ and R is continuous in 0.

(46) If f is partial differentiable on 1st-2nd coordinate in z_0, then there exists R such that $R(0) = 0$ and R is continuous in 0.

(47) If f is partial differentiable on 2nd-1st coordinate in z_0, then there exists R such that $R(0) = 0$ and R is continuous in 0.

(48) If f is partial differentiable on 2nd-2nd coordinate in z_0, then there exists R such that $R(0) = 0$ and R is continuous in 0.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[6] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[7] Noboru Endou, Yasunari Shidama, and Keiichi Miyajima. Partial differentiation on normed linear spaces $\mathcal{R}^n$. *Formalized Mathematics*, 15(**2**):65–72, 2007, doi:10.2478/v10037-007-0008-5.

[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[9] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[10] Jarosław Kotowicz. Properties of real functions. *Formalized Mathematics*, 1(**4**):781–786, 1990.

[11] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[12] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.

[13] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(**4**):797–801, 1990.

[14] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.

[15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[16] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

[17] Bing Xie, Xiquan Liang, and Hongwei Li. Partial differentiation of real binary functions. *Formalized Mathematics*, 16(**4**):333–338, 2008, doi:10.2478/v10037-008-0041-z.

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