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Complex Function Differentiability

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Summary. For a complex valued function defined on its domain in complex numbers the differentiability in a single point and on a subset of the domain is presented. The main elements of differential calculus are developed. The algebraic properties of differential complex functions are shown.

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The articles [17], [18], [3], [5], [4], [8], [2], [7], [11], [6], [16], [12], [19], [9], [10], [1], [14], [15], and [13] provide the notation and terminology for this paper.

For simplicity, we use the following convention: k, n, m denote elements of $\mathbb{N}$, X denotes a set, s_1, s_2 denote complex sequences, Y denotes a subset of $\mathbb{C}$, f, f_1, f_2 denote partial functions from $\mathbb{C}$ to $\mathbb{C}$, r denotes a real number, a, a_1, b, x, x_0, z, z_0 denote complex numbers, and N_1 denotes an increasing sequence of naturals.

Let I be a complex sequence. We say that I is convergent to 0 if and only if:

(Def. 1) I is non-zero and convergent and $\lim I = 0$.

We now state four propositions:

(1) Let r_1 be a sequence of real numbers and c_1 be a complex sequence. If $r_1 = c_1$ and r_1 is convergent, then c_1 is convergent.

(2) If $0 < r$ and for every n holds $s_1(n) = \frac{1}{n+r}$, then s_1 is convergent.

(3) If $0 < r$ and for every n holds $s_1(n) = \frac{1}{n+r}$, then $\lim s_1 = 0$.

(4) If for every n holds $s_1(n) = \frac{1}{n+1}$, then s_1 is convergent and $\lim s_1 = 0$.

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Let us observe that there exists a complex sequence which is convergent to 0.

Let us note that there exists a complex sequence which is constant.

Next we state four propositions:

(5) s_1 is constant iff for all n, m holds $s_1(n) = s_1(m)$.

(6) For every n holds $(s_1 \cdot N_1)(n) = s_1(N_1(n))$.

(7) If s_1 is constant and s_2 is a subsequence of s_1, then s_2 is constant.

(8) If s_1 is constant and s_2 is a subsequence of s_1, then $s_1 = s_2$.

Let s_3 be a constant complex sequence. Note that every subsequence of s_3 is constant.

In the sequel h is a convergent to 0 complex sequence and c is a constant complex sequence.

Let I be a partial function from $\mathbb{C}$ to $\mathbb{C}$. We say that I is rest-like if and only if:

(Def. 2) I is total and for every h holds $h^{-1}\,(I \cdot h)$ is convergent and $\lim(h^{-1}\,(I \cdot h)) = 0$.

Let us mention that there exists a partial function from $\mathbb{C}$ to $\mathbb{C}$ which is rest-like.

A $\mathbb{C}$-rest is a rest-like partial function from $\mathbb{C}$ to $\mathbb{C}$.

Let I be a partial function from $\mathbb{C}$ to $\mathbb{C}$. We say that I is linear if and only if:

(Def. 3) I is total and there exists a such that for every z holds $I_z = a \cdot z$.

One can check that there exists a partial function from $\mathbb{C}$ to $\mathbb{C}$ which is linear.

A $\mathbb{C}$-linear function is a linear partial function from $\mathbb{C}$ to $\mathbb{C}$.

We adopt the following convention: R, R_1, R_2 are $\mathbb{C}$-rests and L, L_1, L_2 are $\mathbb{C}$-linear functions.

Let us consider L_1, L_2. Observe that $L_1 + L_2$ is linear and $L_1 - L_2$ is linear.

The following propositions are true:

(9) For all L_1, L_2 holds $L_1 + L_2$ is a $\mathbb{C}$-linear function and $L_1 - L_2$ is a $\mathbb{C}$-linear function.

(10) For all a, L holds $a\,L$ is a $\mathbb{C}$-linear function.

(11) For all R_1, R_2 holds $R_1 + R_2$ is a $\mathbb{C}$-rest and $R_1 - R_2$ is a $\mathbb{C}$-rest and $R_1\,R_2$ is a $\mathbb{C}$-rest.

(12) $a\,R$ is a $\mathbb{C}$-rest.

(13) $L_1\,L_2$ is rest-like.

(14) $R\,L$ is a $\mathbb{C}$-rest and $L\,R$ is a $\mathbb{C}$-rest.

Let z_0 be a complex number. A subset of $\mathbb{C}$ is called a neighbourhood of z_0 if:

(Def. 4) There exists a real number g such that $0 < g$ and $\{y; y$ ranges over complex numbers: $|y - z_0| < g\} \subseteq$ it.

Next we state three propositions:

(15) For every real number g such that $0 < g$ holds $\{y; y$ ranges over complex numbers: $|y - z_0| < g\}$ is a neighbourhood of z_0.

(16) For every neighbourhood N of z_0 holds $z_0 \in N$.

(17) Let z_0 be a complex number and N_2, N_3 be neighbourhoods of z_0. Then there exists a neighbourhood N of z_0 such that $N \subseteq N_2$ and $N \subseteq N_3$.

Let us consider f and let x_0 be a complex number. We say that f is differentiable in x_0 if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and there exist L, R such that for every complex number x such that $x \in N$ holds $f_x - f_{x_0} = L_{x-x_0} + R_{x-x_0}$.

Let us consider f and let z_0 be a complex number. Let us assume that f is differentiable in z_0. The functor $f'(z_0)$ yielding a complex number is defined by the condition (Def. 6).

(Def. 6) There exists a neighbourhood N of z_0 such that $N \subseteq \operatorname{dom} f$ and there exist L, R such that $f'(z_0) = L_{1_{\mathbb{C}}}$ and for every complex number z such that $z \in N$ holds $f_z - f_{z_0} = L_{z-z_0} + R_{z-z_0}$.

Let us consider f, X. We say that f is differentiable on X if and only if:

(Def. 7) $X \subseteq \operatorname{dom} f$ and for every x such that $x \in X$ holds $f \upharpoonright X$ is differentiable in x.

We now state the proposition

(18) If f is differentiable on X, then X is a subset of $\mathbb{C}$.

Let X be a subset of $\mathbb{C}$. We say that X is closed if and only if:

(Def. 8) For every complex sequence s_3 such that $\operatorname{rng} s_3 \subseteq X$ and s_3 is convergent holds $\lim s_3 \in X$.

Let X be a subset of $\mathbb{C}$. We say that X is open if and only if:

(Def. 9) X^{c} is closed.

Next we state several propositions:

(19) Let X be a subset of $\mathbb{C}$. Suppose X is open. Let z_0 be a complex number. If $z_0 \in X$, then there exists a neighbourhood N of z_0 such that $N \subseteq X$.

(20) Let X be a subset of $\mathbb{C}$. Suppose X is open. Let z_0 be a complex number. Suppose $z_0 \in X$. Then there exists a real number g such that $\{y; y$ ranges over complex numbers: $|y - z_0| < g\} \subseteq X$.

(21) Let X be a subset of $\mathbb{C}$. Suppose that for every complex number z_0 such that $z_0 \in X$ there exists a neighbourhood N of z_0 such that $N \subseteq X$. Then X is open.

(22) Let X be a subset of $\mathbb{C}$. Then X is open if and only if for every complex number x such that $x \in X$ there exists a neighbourhood N of x such that $N \subseteq X$.

(23) Let X be a subset of $\mathbb{C}$, z_0 be an element of $\mathbb{C}$, and r be an element of $\mathbb{R}$. If $X = \{y; y$ ranges over complex numbers: $|y - z_0| < r\}$, then X is open.

(24) Let X be a subset of $\mathbb{C}$, z_0 be an element of $\mathbb{C}$, and r be an element of $\mathbb{R}$. If $X = \{y; y$ ranges over complex numbers: $|y - z_0| \leq r\}$, then X is closed.

Let us note that there exists a subset of $\mathbb{C}$ which is open.

In the sequel Z denotes an open subset of $\mathbb{C}$.

Next we state two propositions:

(25) f is differentiable on Z iff $Z \subseteq \operatorname{dom} f$ and for every x such that $x \in Z$ holds f is differentiable in x.

(26) If f is differentiable on Y, then Y is open.

Let us consider f, X. Let us assume that f is differentiable on X. The functor $f'_{\restriction X}$ yielding a partial function from $\mathbb{C}$ to $\mathbb{C}$ is defined by:

(Def. 10) $\operatorname{dom}(f'_{\restriction X}) = X$ and for every x such that $x \in X$ holds $(f'_{\restriction X})_x = f'(x)$.

The following propositions are true:

(27) Let given f, Z. Suppose $Z \subseteq \operatorname{dom} f$ and there exists a_1 such that $\operatorname{rng} f = \{a_1\}$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\restriction Z})_x = 0_{\mathbb{C}}$.

(28) If s_1 is non-zero, then $s_1 \uparrow k$ is non-zero.

Let us consider h, n. Note that $h \uparrow n$ is convergent to 0.

Let us consider c, n. Note that $c \uparrow n$ is constant.

Next we state a number of propositions:

(29) $(s_1 + s_2) \uparrow k = s_1 \uparrow k + s_2 \uparrow k$.

(30) $(s_1 - s_2) \uparrow k = s_1 \uparrow k - s_2 \uparrow k$.

(31) ${s_1}^{-1} \uparrow k = (s_1 \uparrow k)^{-1}$.

(32) $(s_1\, s_2) \uparrow k = (s_1 \uparrow k)\,(s_2 \uparrow k)$.

(33) Let x_0 be a complex number and N be a neighbourhood of x_0. Suppose f is differentiable in x_0 and $N \subseteq \operatorname{dom} f$. Let given h, c. Suppose $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h + c) \subseteq N$. Then $h^{-1}\,(f \cdot (h + c) - f \cdot c)$ is convergent and $f'(x_0) = \lim(h^{-1}\,(f \cdot (h + c) - f \cdot c))$.

(34) Let given f_1, f_2, x_0. Suppose f_1 is differentiable in x_0 and f_2 is differentiable in x_0. Then $f_1 + f_2$ is differentiable in x_0 and $(f_1 + f_2)'(x_0) = {f_1}'(x_0) + {f_2}'(x_0)$.

(35) Let given f_1, f_2, x_0. Suppose f_1 is differentiable in x_0 and f_2 is differentiable in x_0. Then $f_1 - f_2$ is differentiable in x_0 and $(f_1 - f_2)'(x_0) = {f_1}'(x_0) - {f_2}'(x_0)$.

(36) For all a, f, x_0 such that f is differentiable in x_0 holds $a\,f$ is differentiable in x_0 and $(a\,f)'(x_0) = a \cdot f'(x_0)$.

(37) Let given f_1, f_2, x_0. Suppose f_1 is differentiable in x_0 and f_2 is differentiable in x_0. Then $f_1\,f_2$ is differentiable in x_0 and $(f_1\,f_2)'(x_0) = (f_2)_{x_0} \cdot {f_1}'(x_0) + (f_1)_{x_0} \cdot {f_2}'(x_0)$.

(38) For all f, Z such that $Z \subseteq \mathrm{dom}\, f$ and $f{\restriction}Z = \mathrm{id}_Z$ holds f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\restriction Z})_x = 1_{\mathbb{C}}$.

(39) Let given f_1, f_2, Z. Suppose $Z \subseteq \mathrm{dom}(f_1 + f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1 + f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $((f_1 + f_2)'_{\restriction Z})_x = {f_1}'(x) + {f_2}'(x)$.

(40) Let given f_1, f_2, Z. Suppose $Z \subseteq \mathrm{dom}(f_1 - f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1 - f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $((f_1 - f_2)'_{\restriction Z})_x = {f_1}'(x) - {f_2}'(x)$.

(41) Let given a, f, Z. Suppose $Z \subseteq \mathrm{dom}(a\,f)$ and f is differentiable on Z. Then $a\,f$ is differentiable on Z and for every x such that $x \in Z$ holds $((a\,f)'_{\restriction Z})_x = a \cdot f'(x)$.

(42) Let given f_1, f_2, Z. Suppose $Z \subseteq \mathrm{dom}(f_1\,f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1\,f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $((f_1\,f_2)'_{\restriction Z})_x = (f_2)_x \cdot {f_1}'(x) + (f_1)_x \cdot {f_2}'(x)$.

(43) If $Z \subseteq \mathrm{dom}\, f$ and f is a constant on Z, then f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\restriction Z})_x = 0_{\mathbb{C}}$.

(44) Suppose $Z \subseteq \mathrm{dom}\, f$ and for every x such that $x \in Z$ holds $f_x = a \cdot x + b$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\restriction Z})_x = a$.

(45) For every complex number x_0 such that f is differentiable in x_0 holds f is continuous in x_0.

(46) If f is differentiable on X, then f is continuous on X.

(47) If f is differentiable on X and $Z \subseteq X$, then f is differentiable on Z.

(48) If s_1 is convergent, then $|s_1|$ is convergent.

(49) If f is differentiable in x_0, then there exists R such that $R_{0_{\mathbb{C}}} = 0_{\mathbb{C}}$ and R is continuous in $0_{\mathbb{C}}$.

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