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Several Integrability Formulas of Special Functions. Part II

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Summary. In this article, we give several differentiation and integrability formulas of special and composite functions including the trigonometric function, the hyperbolic function and the polynomial function [3].

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The articles [10], [23], [19], [21], [22], [1], [8], [15], [9], [2], [4], [17], [5], [13], [16], [14], [18], [7], [12], [20], [6], and [11] provide the terminology and notation for this paper.

1. Differentiation Formulas

For simplicity, we adopt the following rules: r, x, a, b denote real numbers, n, m denote elements of $\mathbb{N}$, A denotes a closed-interval subset of $\mathbb{R}$, and Z denotes an open subset of $\mathbb{R}$.

One can prove the following propositions:

(1)(i) $(\frac{1}{2}\square+0) - \frac{1}{4}\,((\text{the function sin})\ \cdot(2\square+0))$ is differentiable on $\mathbb{R}$, and

(ii) for every x holds $((\frac{1}{2}\square+0) - \frac{1}{4}\,((\text{the function sin})\ \cdot(2\square+0)))'_{\restriction\mathbb{R}}(x) = (\sin x)^{\mathbf{2}}$.

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(2)(i) $(\frac{1}{2}\square+0) + \frac{1}{4}\,((\text{the function sin})\ \cdot(2\square+0))$ is differentiable on $\mathbb{R}$, and

(ii) for every x holds $((\frac{1}{2}\square+0) + \frac{1}{4}\,((\text{the function sin})\ \cdot(2\square+0)))'_{\restriction\mathbb{R}}(x) = (\cos x)^{\mathbf{2}}$.

(3) $\frac{1}{n+1}\,((\square^{n+1}) \cdot (\text{the function sin}))$ is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{n+1}\,(\text{the function sin})^{n+1})'_{\restriction\mathbb{R}}(x) = (\sin x)^n \cdot \cos x$.

(4)(i) $(-\frac{1}{n+1})\,((\square^{n+1}) \cdot (\text{the function cos}))$ is differentiable on $\mathbb{R}$, and

(ii) for every x holds $((-\frac{1}{n+1})\,(\text{the function cos})^{n+1})'_{\restriction\mathbb{R}}(x) = (\cos x)^n \cdot \sin x$.

(5) Suppose $m + n \neq 0$ and $m - n \neq 0$. Then

(i) $\frac{1}{2\cdot(m+n)}\,((\text{the function sin})\ \cdot((m+n)\square+0)) + \frac{1}{2\cdot(m-n)}\,((\text{the function sin})\ \cdot((m-n)\square+0))$ is differentiable on $\mathbb{R}$, and

(ii) for every x holds $(\frac{1}{2\cdot(m+n)}\,((\text{the function sin})\ \cdot((m+n)\square+0)) + \frac{1}{2\cdot(m-n)}\,((\text{the function sin})\ \cdot((m-n)\square+0)))'_{\restriction\mathbb{R}}(x) = \cos(m \cdot x) \cdot \cos(n \cdot x)$.

(6) Suppose $m + n \neq 0$ and $m - n \neq 0$. Then

(i) $\frac{1}{2\cdot(m-n)}\,((\text{the function sin})\ \cdot((m-n)\square+0)) - \frac{1}{2\cdot(m+n)}\,((\text{the function sin})\ \cdot((m+n)\square+0))$ is differentiable on $\mathbb{R}$, and

(ii) for every x holds $(\frac{1}{2\cdot(m-n)}\,((\text{the function sin})\ \cdot((m-n)\square+0)) - \frac{1}{2\cdot(m+n)}\,((\text{the function sin})\ \cdot((m+n)\square+0)))'_{\restriction\mathbb{R}}(x) = \sin(m \cdot x) \cdot \sin(n \cdot x)$.

(7) Suppose $m + n \neq 0$ and $m - n \neq 0$. Then

(i) $-\frac{1}{2\cdot(m+n)}\,((\text{the function cos}) \cdot ((m+n)\square+0)) - \frac{1}{2\cdot(m-n)}\,((\text{the function cos})\ \cdot((m-n)\square+0))$ is differentiable on $\mathbb{R}$, and

(ii) for every x holds $(-\frac{1}{2\cdot(m+n)}\,((\text{the function cos}) \cdot ((m+n)\square+0)) - \frac{1}{2\cdot(m-n)}\,((\text{the function cos})\ \cdot((m-n)\square+0)))'_{\restriction\mathbb{R}}(x) = \sin(m \cdot x) \cdot \cos(n \cdot x)$.

(8) Suppose $n \neq 0$. Then

(i) $\frac{1}{n^{\mathbf{2}}}\,((\text{the function sin})\ \cdot(n\square+0)) - (\frac{1}{n}\square+0)\,((\text{the function cos})\ \cdot(n\square+0))$ is differentiable on $\mathbb{R}$, and

(ii) for every x holds $(\frac{1}{n^{\mathbf{2}}}\,((\text{the function sin})\ \cdot(n\square+0)) - (\frac{1}{n}\square+0)\,((\text{the function cos})\ \cdot(n\square+0)))'_{\restriction\mathbb{R}}(x) = x \cdot \sin(n \cdot x)$.

(9) Suppose $n \neq 0$. Then

(i) $\frac{1}{n^{\mathbf{2}}}\,((\text{the function cos})\ \cdot(n\square+0)) + (\frac{1}{n}\square+0)\,((\text{the function sin})\ \cdot(n\square+0))$ is differentiable on $\mathbb{R}$, and

(ii) for every x holds $(\frac{1}{n^{\mathbf{2}}}\,((\text{the function cos})\ \cdot(n\square+0)) + (\frac{1}{n}\square+0)\,((\text{the function sin})\ \cdot(n\square+0)))'_{\restriction\mathbb{R}}(x) = x \cdot \cos(n \cdot x)$.

(10)(i) $(1\square+0)\,(\text{the function cosh}) - \text{the function sinh}$ is differentiable on $\mathbb{R}$, and

(ii) for every x holds $((1\square+0)\,(\text{the function cosh}) - \text{the function sinh})'_{\restriction\mathbb{R}}(x) = x \cdot \sinh x$.

(11)(i) $(1\square+0)\,(\text{the function sinh}) - \text{the function cosh}$ is differentiable on $\mathbb{R}$, and

(ii) for every x holds $((1\square{+}0)\,(\text{the function sinh}) - \text{the function cosh})'_{\restriction\mathbb{R}}(x) = x \cdot \cosh x$.

(12) If $a \cdot (n+1) \neq 0$, then $\frac{1}{a\cdot(n+1)}\,(a\square{+}b)^{n+1}$ is differentiable on $\mathbb{R}$ and for every x holds $(\frac{1}{a\cdot(n+1)}\,(a\square{+}b)^{n+1})'_{\restriction\mathbb{R}}(x) = (a \cdot x + b)^n$.

2. Integrability Formulas

Next we state a number of propositions:

(13) $\int\limits_{A}(\text{the function sin})^{\mathbf{2}}(x)dx = \frac{1}{2} \cdot \sup A - \frac{1}{4} \cdot \sin(2 \cdot \sup A) - (\frac{1}{2} \cdot \inf A - \frac{1}{4} \cdot \sin(2 \cdot \inf A))$.

(14) $\int\limits_{[0,\pi]}(\text{the function sin})^{\mathbf{2}}(x)dx = \frac{\pi}{2}$.

(15) $\int\limits_{[0,2\cdot\pi]}(\text{the function sin})^{\mathbf{2}}(x)dx = \pi$.

(16) $\int\limits_{A}(\text{the function cos})^{\mathbf{2}}(x)dx = (\frac{1}{2} \cdot \sup A + \frac{1}{4} \cdot \sin(2 \cdot \sup A)) - (\frac{1}{2} \cdot \inf A + \frac{1}{4} \cdot \sin(2 \cdot \inf A))$.

(17) $\int\limits_{[0,\pi]}(\text{the function cos})^{\mathbf{2}}(x)dx = \frac{\pi}{2}$.

(18) $\int\limits_{[0,2\cdot\pi]}(\text{the function cos})^{\mathbf{2}}(x)dx = \pi$.

(19) $\int\limits_{A}((\text{the function sin})^{n}\,(\text{the function cos}))(x)dx = \frac{1}{n+1} \cdot (\sin \sup A)^{n+1} - \frac{1}{n+1} \cdot (\sin \inf A)^{n+1}$.

(20) $\int\limits_{[0,\pi]}((\text{the function sin})^{n}\,(\text{the function cos}))(x)dx = 0$.

(21) $\int\limits_{[0,2\cdot\pi]}((\text{the function sin})^{n}\,(\text{the function cos}))(x)dx = 0$.

(22) $\int\limits_{A}((\text{the function cos})^{n}\,(\text{the function sin}))(x)dx = (-\frac{1}{n+1}) \cdot (\cos \sup A)^{n+1} - (-\frac{1}{n+1}) \cdot (\cos \inf A)^{n+1}$.

(23) $\displaystyle\int_{[0,2\cdot\pi]} ((\text{the function } \cos)^n \, (\text{the function } \sin))(x)dx = 0.$

(24) $\displaystyle\int_{[-\frac{\pi}{2},\frac{\pi}{2}]} ((\text{the function } \cos)^n \, (\text{the function } \sin))(x)dx = 0.$

(25) Suppose $m+n \neq 0$ and $m-n \neq 0$. Then $\displaystyle\int_A (((\text{the function } \cos)\cdot(m\square{+}0))\,((\text{the function } \cos)\cdot(n\square{+}0)))(x)dx = (\frac{1}{2\cdot(m+n)}\cdot\sin((m+n)\cdot\sup A)+\frac{1}{2\cdot(m-n)}\cdot\sin((m-n)\cdot\sup A)) - (\frac{1}{2\cdot(m+n)}\cdot\sin((m+n)\cdot\inf A)+\frac{1}{2\cdot(m-n)}\cdot\sin((m-n)\cdot\inf A)).$

(26) Suppose $m+n \neq 0$ and $m-n \neq 0$. Then $\displaystyle\int_A (((\text{the function } \sin)\cdot(m\square{+}0))\,((\text{the function } \sin)\cdot(n\square{+}0)))(x)dx = \frac{1}{2\cdot(m-n)}\cdot\sin((m-n)\cdot\sup A)-\frac{1}{2\cdot(m+n)}\cdot\sin((m+n)\cdot\sup A) - (\frac{1}{2\cdot(m-n)}\cdot\sin((m-n)\cdot\inf A)-\frac{1}{2\cdot(m+n)}\cdot\sin((m+n)\cdot\inf A)).$

(27) Suppose $m+n \neq 0$ and $m-n \neq 0$. Then $\displaystyle\int_A (((\text{the function } \sin)\cdot(m\square{+}0))\,((\text{the function } \cos)\cdot(n\square{+}0)))(x)dx = -\frac{1}{2\cdot(m+n)}\cdot\cos((m+n)\cdot\sup A)-\frac{1}{2\cdot(m-n)}\cdot\cos((m-n)\cdot\sup A) - (-\frac{1}{2\cdot(m+n)}\cdot\cos((m+n)\cdot\inf A)-\frac{1}{2\cdot(m-n)}\cdot\cos((m-n)\cdot\inf A)).$

(28) If $n \neq 0$, then $\displaystyle\int_A ((1\square{+}0)\,((\text{the function } \sin)\cdot(n\square{+}0)))(x)dx = \frac{1}{n^{\mathbf{2}}}\cdot \sin(n\cdot\sup A)-\frac{1}{n}\cdot\sup A\cdot\cos(n\cdot\sup A)-(\frac{1}{n^{\mathbf{2}}}\cdot\sin(n\cdot\inf A)-\frac{1}{n}\cdot\inf A\cdot \cos(n\cdot\inf A)).$

(29) If $n \neq 0$, then $\displaystyle\int_A ((1\square{+}0)\,((\text{the function } \cos)\cdot(n\square{+}0)))(x)dx = (\frac{1}{n^{\mathbf{2}}}\cdot \cos(n\cdot\sup A)+\frac{1}{n}\cdot\sup A\cdot\sin(n\cdot\sup A))-(\frac{1}{n^{\mathbf{2}}}\cdot\cos(n\cdot\inf A)+\frac{1}{n}\cdot\inf A\cdot \sin(n\cdot\inf A)).$

(30) $\displaystyle\int_A ((1\square{+}0)\,(\text{the function } \sinh))(x)dx = \sup A\cdot\cosh\sup A-\sinh\sup A - (\inf A\cdot\cosh\inf A-\sinh\inf A).$

(31) $\displaystyle\int_A ((1\square{+}0)\,(\text{the function } \cosh))(x)dx = \sup A\cdot\sinh\sup A-\cosh\sup A - (\inf A\cdot\sinh\inf A-\cosh\inf A).$

(32) If $a \cdot (n+1) \neq 0$, then $\int\limits_A (a\square{+}b)^n(x)dx = \frac{1}{a \cdot (n+1)} \cdot (a \cdot \sup A + b)^{n+1} - \frac{1}{a \cdot (n+1)} \cdot (a \cdot \inf A + b)^{n+1}$.

3. Addenda

In the sequel f, f_1, f_2, f_3, g are partial functions from $\mathbb{R}$ to $\mathbb{R}$.

The following propositions are true:

(33) If $Z \subseteq \mathrm{dom}(\frac{1}{2}\, f)$ and $f = \square^2$, then $\frac{1}{2}\, f$ is differentiable on Z and for every x such that $x \in Z$ holds $(\frac{1}{2}\, f)'_{\restriction Z}(x) = x$.

(34) If $A \subseteq Z = \mathrm{dom}(\frac{1}{2}\, (\square^2))$, then $\int\limits_A \mathrm{id}_Z(x)dx = \frac{1}{2} \cdot (\sup A)^2 - \frac{1}{2} \cdot (\inf A)^2$.

(35) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $g(x) = x$ and $g(x) \neq 0$ and $f(x) = -\frac{1}{x^{\mathbf{2}}}$ and $Z = \mathrm{dom}\, g$ and $\mathrm{dom}\, f = Z$ and $f{\restriction}A$ is continuous. Then $\int\limits_A f(x)dx = (\sup A)^{-1} - (\inf A)^{-1}$.

(36) Suppose that

(i) $A \subseteq Z$,

(ii) $f_1 = \square^2$,

(iii) for every x such that $x \in Z$ holds $f_2(x) = 1$ and $x \neq 0$ and $f(x) = \frac{2 \cdot x}{(1+x^{\mathbf{2}})^{\mathbf{2}}}$,

(iv) $\mathrm{dom}(\frac{f_1}{f_2+f_1}) = Z$,

(v) $Z = \mathrm{dom}\, f$, and

(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (\frac{f_1}{f_2+f_1})(\sup A) - (\frac{f_1}{f_2+f_1})(\inf A)$.

(37) Suppose $Z \subseteq \mathrm{dom}(\text{(the function tan)+(the function sec)})$ and for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$. Then

(i) (the function tan)+(the function sec) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function tan)+(the function sec))$'_{\restriction Z}(x) = \frac{1}{1-\sin x}$.

(38) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$ and $f(x) = \frac{1}{1-\sin x}$,

(iii) dom((the function tan)+(the function sec)) $= Z$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (\tan\sup A + \sec\sup A) - (\tan\inf A + \sec\inf A)$.

(39) Suppose $Z \subseteq \mathrm{dom}((\text{the function tan}) - (\text{the function sec}))$ and for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$. Then

(i) (the function tan)$-$(the function sec) is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ((the function tan)$-$(the function sec))$'_{\restriction Z}(x) = \frac{1}{1+\sin x}$.

(40) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$ and $f(x) = \frac{1}{1+\sin x}$,

(iii) dom((the function tan)$-$(the function sec)) $= Z$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = \tan\sup A - \sec\sup A - (\tan\inf A - \sec\inf A)$.

(41) Suppose $Z \subseteq \mathrm{dom}(-\text{the function cot} + \text{the function cosec})$ and for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$. Then

(i) $-$the function cot $+$ the function cosec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ($-$the function cot $+$ the function cosec)$'_{\restriction Z}(x) = \frac{1}{1+\cos x}$.

(42) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$ and $f(x) = \frac{1}{1+\cos x}$,

(iii) dom($-$the function cot $+$ the function cosec) $= Z$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = (-\cot\sup A + \operatorname{cosec}\sup A) - (-\cot\inf A + \operatorname{cosec}\inf A)$.

(43) Suppose $Z \subseteq \mathrm{dom}(-\text{the function cot} - \text{the function cosec})$ and for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$. Then

(i) $-$the function cot $-$ the function cosec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds ($-$the function cot $-$ the function cosec)$'_{\restriction Z}(x) = \frac{1}{1-\cos x}$.

(44) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$ and $f(x) = \frac{1}{1-\cos x}$,

(iii) dom($-$the function cot $-$ the function cosec) $= Z$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = -\cot \sup A - \operatorname{cosec} \sup A - (-\cot \inf A - \operatorname{cosec} \inf A)$.

(45) Suppose that
(i) $A \subseteq Z$,
(ii) $Z \subseteq]-1, 1[$,
(iii) for every x such that $x \in Z$ holds $f(x) = \frac{1}{1+x^{\mathbf{2}}}$,
(iv) dom (the function arctan) $= Z$,
(v) $Z = \operatorname{dom} f$, and
(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = \arctan \sup A - \arctan \inf A$.

(46) Suppose that
(i) $A \subseteq Z$,
(ii) $Z \subseteq]-1, 1[$,
(iii) for every x such that $x \in Z$ holds $f(x) = \frac{r}{1+x^{\mathbf{2}}}$,
(iv) dom(r the function arctan) $= Z$,
(v) $Z = \operatorname{dom} f$, and
(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = r \cdot \arctan \sup A - r \cdot \arctan \inf A$.

(47) Suppose that
(i) $A \subseteq Z$,
(ii) $Z \subseteq]-1, 1[$,
(iii) for every x such that $x \in Z$ holds $f(x) = -\frac{1}{1+x^{\mathbf{2}}}$,
(iv) dom (the function arccot) $= Z$,
(v) $Z = \operatorname{dom} f$, and
(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = \operatorname{arccot} \sup A - \operatorname{arccot} \inf A$.

(48) Suppose that
(i) $A \subseteq Z$,
(ii) $Z \subseteq]-1, 1[$,
(iii) for every x such that $x \in Z$ holds $f(x) = -\frac{r}{1+x^{\mathbf{2}}}$,
(iv) dom(r the function arccot) $= Z$,
(v) $Z = \operatorname{dom} f$, and
(vi) $f{\restriction}A$ is continuous.

Then $\int\limits_{A} f(x)dx = r \cdot \operatorname{arccot} \sup A - r \cdot \operatorname{arccot} \inf A$.

(49) Suppose $Z \subseteq \operatorname{dom}((\operatorname{id}_Z +$ the function cot$)-$the function cosec) and for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$. Then

(i) $(\mathrm{id}_Z + \text{the function cot})-\text{the function cosec}$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\mathrm{id}_Z+\text{the function cot})-\text{the function cosec})'_{\restriction Z}(x) = \frac{\cos x}{1+\cos x}$.

(50) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$ and $f(x) = \frac{\cos x}{1+\cos x}$,
(iii) $\mathrm{dom}((\mathrm{id}_Z + \text{the function cot})-\text{the function cosec}) = Z$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (\sup A+\cot\sup A) - \mathrm{cosec}\sup A - ((\inf A+\cot\inf A) - \mathrm{cosec}\inf A)$.

(51) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z + \text{the function cot}+\text{the function cosec})$ and for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$. Then
(i) $\mathrm{id}_Z + \text{the function cot}+\text{the function cosec}$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z + \text{the function cot}+\text{the function cosec})'_{\restriction Z}(x) = \frac{\cos x}{\cos x - 1}$.

(52) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $1 + \cos x \neq 0$ and $1 - \cos x \neq 0$ and $f(x) = \frac{\cos x}{\cos x-1}$,
(iii) $\mathrm{dom}(\mathrm{id}_Z + \text{the function cot}+\text{the function cosec}) = Z$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = (\sup A + \cot\sup A + \mathrm{cosec}\sup A) - (\inf A + \cot\inf A + \mathrm{cosec}\inf A)$.

(53) Suppose $Z \subseteq \mathrm{dom}((\mathrm{id}_Z - \text{the function tan})+\text{the function sec})$ and for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$. Then
(i) $(\mathrm{id}_Z - \text{the function tan})+\text{the function sec}$ is differentiable on Z, and
(ii) for every x such that $x \in Z$ holds $((\mathrm{id}_Z-\text{the function tan})+\text{the function sec})'_{\restriction Z}(x) = \frac{\sin x}{\sin x+1}$.

(54) Suppose that
(i) $A \subseteq Z$,
(ii) for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$ and $f(x) = \frac{\sin x}{1+\sin x}$,
(iii) $Z \subseteq \mathrm{dom}((\mathrm{id}_Z - \text{the function tan})+\text{the function sec})$,
(iv) $Z = \mathrm{dom}\, f$, and
(v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = ((\sup A - \tan\sup A) + \sec\sup A) - ((\inf A - \tan\inf A) + \sec\inf A)$.

(55) Suppose $Z \subseteq \mathrm{dom}(\mathrm{id}_Z - \text{the function tan} - \text{the function sec})$ and for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$. Then

(i) $\mathrm{id}_Z -$ the function tan$-$the function sec is differentiable on Z, and

(ii) for every x such that $x \in Z$ holds $(\mathrm{id}_Z - \text{the function tan} - \text{the function sec})'_{\restriction Z}(x) = \frac{\sin x}{\sin x - 1}$.

(56) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $1 + \sin x \neq 0$ and $1 - \sin x \neq 0$ and $f(x) = \frac{\sin x}{\sin x - 1}$,

(iii) $Z \subseteq \mathrm{dom}(\mathrm{id}_Z - \text{the function tan} - \text{the function sec})$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = \sup A - \tan\sup A - \sec\sup A - (\inf A - \tan\inf A - \sec\inf A)$.

(57) Suppose $Z \subseteq \mathrm{dom}((\text{the function tan}) - \mathrm{id}_Z)$. Then (the function tan)$-\mathrm{id}_Z$ is differentiable on Z and for every x such that $x \in Z$ holds $((\text{the function tan}) - \mathrm{id}_Z)'_{\restriction Z}(x) = (\tan x)^{\mathbf{2}}$.

(58) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $(\text{the function cos})(x) > 0$ and $f(x) = (\tan x)^{\mathbf{2}}$,

(iii) $Z \subseteq \mathrm{dom}((\text{the function tan}) - \mathrm{id}_Z)$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = \tan\sup A - \sup A - (\tan\inf A - \inf A)$.

(59) Suppose $Z \subseteq \mathrm{dom}(-\text{the function cot} - \mathrm{id}_Z)$. Then $-$the function cot $-\,\mathrm{id}_Z$ is differentiable on Z and for every x such that $x \in Z$ holds $(-\text{the function cot} - \mathrm{id}_Z)'_{\restriction Z}(x) = (\cot x)^{\mathbf{2}}$.

(60) Suppose that

(i) $A \subseteq Z$,

(ii) for every x such that $x \in Z$ holds $(\text{the function sin})(x) > 0$ and $f(x) = (\cot x)^{\mathbf{2}}$,

(iii) $Z \subseteq \mathrm{dom}(-\text{the function cot} - \mathrm{id}_Z)$,

(iv) $Z = \mathrm{dom}\, f$, and

(v) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = -\cot \sup A - \sup A - (-\cot \inf A - \inf A)$.

(61) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{1}{(\cos x)^{\mathbf{2}}}$ and $\cos x \neq 0$ and dom (the function tan) $= Z = \mathrm{dom}\, f$ and $f{\restriction}A$ is continuous. Then $\int\limits_A f(x)dx = \tan \sup A - \tan \inf A$.

(62) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $f(x) = -\frac{1}{(\sin x)^{\mathbf{2}}}$ and $\sin x \neq 0$ and dom (the function cot) $= Z = \mathrm{dom}\, f$ and $f{\restriction}A$ is continuous. Then $\int\limits_A f(x)dx = \cot \sup A - \cot \inf A$.

(63) Suppose $A \subseteq Z$ and for every x such that $x \in Z$ holds $f(x) = \frac{\sin x - (\cos x)^{\mathbf{2}}}{(\cos x)^{\mathbf{2}}}$ and $Z \subseteq$ dom((the function sec)$-\mathrm{id}_Z$) and $Z = \mathrm{dom}\, f$ and $f{\restriction}A$ is continuous. Then $\int\limits_A f(x)dx = \sec \sup A - \sup A - (\sec \inf A - \inf A)$.

(64) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $f(x) = \frac{\cos x - (\sin x)^{\mathbf{2}}}{(\sin x)^{\mathbf{2}}}$,
- (iii) $Z \subseteq$ dom($-$the function cosec $-\ \mathrm{id}_Z$),
- (iv) $Z = \mathrm{dom}\, f$, and
- (v) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = -\mathrm{cosec} \sup A - \sup A - (-\mathrm{cosec} \inf A - \inf A)$.

The following propositions are true:

(65) Suppose that
- (i) $A \subseteq Z$,
- (ii) for every x such that $x \in Z$ holds $\sin x > 0$,
- (iii) $Z \subseteq$ dom((the function ln) $\cdot$(the function sin)),
- (iv) $Z =$ dom (the function cot), and
- (v) (the function cot)$\restriction A$ is continuous.

Then $\int\limits_A$ (the function cot)$(x)dx = \ln \sin \sup A - \ln \sin \inf A$.

(66) Suppose that
- (i) $A \subseteq Z$,
- (ii) $Z \subseteq]{-1}, 1[$,
- (iii) for every x such that $x \in Z$ holds $f(x) = \frac{\arcsin x}{\sqrt{1-x^{\mathbf{2}}}}$,
- (iv) $Z \subseteq$ dom($\frac{1}{2}$ (the function arcsin)^2),
- (v) $Z = \mathrm{dom}\, f$, and
- (vi) $f{\restriction}A$ is continuous.

Then $\int\limits_A f(x)dx = \frac{1}{2} \cdot (\arcsin \sup A)^2 - \frac{1}{2} \cdot (\arcsin \inf A)^2$.

(67) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) for every x such that $x \in Z$ holds $f(x) = -\frac{\arccos x}{\sqrt{1-x^{\mathbf{2}}}}$,

(iv) $Z \subseteq \mathrm{dom}(\frac{1}{2}$ (the function arccos)$^2)$,

(v) $Z = \mathrm{dom}\, f$, and

(vi) $f \restriction A$ is continuous.

Then $\int\limits_A f(x)dx = \frac{1}{2} \cdot (\arccos \sup A)^2 - \frac{1}{2} \cdot (\arccos \inf A)^2$.

(68) $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f(x) > 0$ and $x \neq 0$ and dom (the function arcsin) $= Z \subseteq \mathrm{dom}(\mathrm{id}_Z$ (the function arcsin)$+f^{\frac{1}{2}})$.

(69) Suppose that $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = a^{\mathbf{2}}$ and $f(x) > 0$ and $f_3(x) = \frac{x}{a}$ and $-1 < f_3(x) < 1$ and $x \neq 0$ and $a > 0$ and dom((the function arcsin) $\cdot f_3) = Z \subseteq \mathrm{dom}(\mathrm{id}_Z$ ((the function arcsin) $\cdot f_3) + (\square^{\frac{1}{2}}) \cdot f)$ and ((the function arcsin) $\cdot f_3) \restriction A$ is continuous. Then $\int\limits_A$((the function arcsin) $\cdot f_3)(x)dx =$ $(\sup A \cdot \arcsin(\frac{\sup A}{a}) + f(\sup A)^{\frac{1}{2}}) - (\inf A \cdot \arcsin(\frac{\inf A}{a}) + f(\inf A)^{\frac{1}{2}})$.

(70) Suppose that $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = 1$ and $f(x) > 0$ and $x \neq 0$ and dom (the function arccos) $= Z \subseteq \mathrm{dom}(\mathrm{id}_Z$ (the function arccos)$-(\square^{\frac{1}{2}}) \cdot f)$. Then $\int\limits_A$(the function arccos)$(x)dx = \sup A \cdot \arccos \sup A - f(\sup A)^{\frac{1}{2}} -$ $(\inf A \cdot \arccos \inf A - f(\inf A)^{\frac{1}{2}})$.

(71) Suppose that $A \subseteq Z \subseteq]-1, 1[$ and $f = f_1 - f_2$ and $f_2 = \square^2$ and for every x such that $x \in Z$ holds $f_1(x) = a^{\mathbf{2}}$ and $f(x) > 0$ and $f_3(x) = \frac{x}{a}$ and $-1 < f_3(x) < 1$ and $x \neq 0$ and $a > 0$ and dom((the function arccos) $\cdot f_3) = Z = \mathrm{dom}(\mathrm{id}_Z$ ((the function arccos) $\cdot f_3) - (\square^{\frac{1}{2}}) \cdot f)$ and ((the function arccos) $\cdot f_3) \restriction A$ is continuous. Then $\int\limits_A$((the function arccos) $\cdot f_3)(x)dx =$ $\sup A \cdot \arccos(\frac{\sup A}{a}) - f(\sup A)^{\frac{1}{2}} - (\inf A \cdot \arccos(\frac{\inf A}{a}) - f(\inf A)^{\frac{1}{2}})$.

(72) Suppose that

(i) $A \subseteq Z$,

(ii) $Z \subseteq]-1, 1[$,

(iii) $f_2 = \square^2$,

(iv) for every x such that $x \in Z$ holds $f_1(x) = 1$,
(v) $Z = \mathrm{dom}\,(\text{the function arctan})$, and
(vi) $Z = \mathrm{dom}(\mathrm{id}_Z \text{ the function arctan} - \frac{1}{2}\,((\text{the function ln})\ \cdot (f_1 + f_2)))$.

Then $\int\limits_A (\text{the function arctan})(x)dx = \sup A \cdot \operatorname{arctan} \sup A - \frac{1}{2} \cdot \ln(1 + (\sup A)^2) - (\inf A \cdot \operatorname{arctan} \inf A - \frac{1}{2} \cdot \ln(1 + (\inf A)^2))$.

(73) Suppose that
(i) $A \subseteq Z$,
(ii) $Z \subseteq]-1, 1[$,
(iii) $f_2 = \Box^2$,
(iv) for every x such that $x \in Z$ holds $f_1(x) = 1$,
(v) $\mathrm{dom}\,(\text{the function arccot}) = Z$, and
(vi) $Z = \mathrm{dom}(\mathrm{id}_Z \text{ the function arccot} + \frac{1}{2}\,((\text{the function ln})\ \cdot (f_1 + f_2)))$.

Then $\int\limits_A (\text{the function arccot})(x)dx = (\sup A \cdot \operatorname{arccot} \sup A + \frac{1}{2} \cdot \ln(1 + (\sup A)^2)) - (\inf A \cdot \operatorname{arccot} \inf A + \frac{1}{2} \cdot \ln(1 + (\inf A)^2))$.

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