FORMALIZED MATHEMATICS
Vol. 16, No. 3, Pages 269–275, 2008

Linear Map of Matrices

Karol Pąk
Institute of Computer Science
University of Białystok
Poland

Summary. The paper is concerned with a generalization of concepts introduced in [13], i.e. introduced are matrices of linear transformations over a finite-dimensional vector space. Introduced are linear transformations over a finite-dimensional vector space depending on a given matrix of the transformation. Finally, I prove that the rank of linear transformations over a finite-dimensional vector space is the same as the rank of the matrix of that transformation.

MML identifier: `MATRLIN2`, version: `7.9.03 4.104.1021`

The notation and terminology used here are introduced in the following papers: [24], [2], [3], [9], [25], [6], [8], [7], [4], [23], [19], [12], [10], [27], [28], [26], [22], [20], [18], [29], [5], [15], [13], [17], [11], [14], [21], [1], and [16].

1. Preliminaries

We adopt the following rules: i, j, m, n are natural numbers, K is a field, and a is an element of K.

One can prove the following propositions:

(1) Let V be a vector space over K, W_1, W_2, W_{12} be subspaces of V, and U_1, U_2 be subspaces of W_{12}. If $U_1 = W_1$ and $U_2 = W_2$, then $W_1 \cap W_2 = U_1 \cap U_2$ and $W_1 + W_2 = U_1 + U_2$.

(2) Let V be a vector space over K and W_1, W_2 be subspaces of V. Suppose $W_1 \cap W_2 = \mathbf{0}_V$. Let B_1 be a linearly independent subset of W_1 and B_2 be a linearly independent subset of W_2. Then $B_1 \cup B_2$ is a linearly independent subset of $W_1 + W_2$.

ISSN 1426–2630(p), 1898-9934(e)

(3) Let V be a vector space over K and W_1, W_2 be subspaces of V. Suppose $W_1 \cap W_2 = \mathbf{0}_V$. Let B_1 be a basis of W_1 and B_2 be a basis of W_2. Then $B_1 \cup B_2$ is a basis of $W_1 + W_2$.

(4) For every finite dimensional vector space V over K holds every ordered basis of Ω_V is an ordered basis of V.

(5) Let V_1 be a vector space over K and A be a finite subset of V_1. If $\dim(\mathrm{Lin}(A)) = \mathrm{card}\, A$, then A is linearly independent.

(6) For every vector space V over K and for every finite subset A of V holds $\dim(\mathrm{Lin}(A)) \leq \mathrm{card}\, A$.

2. More on the Product of Finite Sequence of Scalars and Vectors

For simplicity, we follow the rules: V_1, V_2, V_3 are finite dimensional vector spaces over K, f is a function from V_1 into V_2, b_1, b'_1 are ordered bases of V_1, B_1 is a finite sequence of elements of V_1, b_2 is an ordered basis of V_2, B_2 is a finite sequence of elements of V_2, B_3 is a finite sequence of elements of V_3, v_1, w_1 are elements of V_1, R, R_1, R_2 are finite sequences of elements of V_1, and p, p_1, p_2 are finite sequences of elements of K.

We now state a number of propositions:

(7) $\mathrm{lmlt}(p_1 + p_2, R) = \mathrm{lmlt}(p_1, R) + \mathrm{lmlt}(p_2, R)$.

(8) $\mathrm{lmlt}(p, R_1 + R_2) = \mathrm{lmlt}(p, R_1) + \mathrm{lmlt}(p, R_2)$.

(9) If $\mathrm{len}\, p_1 = \mathrm{len}\, R_1$ and $\mathrm{len}\, p_2 = \mathrm{len}\, R_2$, then $\mathrm{lmlt}(p_1 \frown p_2, R_1 \frown R_2) = (\mathrm{lmlt}(p_1, R_1)) \frown \mathrm{lmlt}(p_2, R_2)$.

(10) If $\mathrm{len}\, R_1 = \mathrm{len}\, R_2$, then $\sum(R_1 + R_2) = (\sum R_1) + \sum R_2$.

(11) $\sum \mathrm{lmlt}(\mathrm{len}\, R \mapsto a, R) = a \cdot \sum R$.

(12) $\sum \mathrm{lmlt}(p, \mathrm{len}\, p \mapsto v_1) = (\sum p) \cdot v_1$.

(13) $\sum \mathrm{lmlt}(a \cdot p, R) = a \cdot \sum \mathrm{lmlt}(p, R)$.

(14) Let B_1 be a finite sequence of elements of V_1, W_1 be a subspace of V_1, and B_2 be a finite sequence of elements of W_1. If $B_1 = B_2$, then $\mathrm{lmlt}(p, B_1) = \mathrm{lmlt}(p, B_2)$.

(15) Let B_1 be a finite sequence of elements of V_1, W_1 be a subspace of V_1, and B_2 be a finite sequence of elements of W_1. If $B_1 = B_2$, then $\sum B_1 = \sum B_2$.

(16) If $i \in \mathrm{dom}\, R$, then $\sum \mathrm{lmlt}(\mathrm{Line}(I_K^{\mathrm{len}\, R \times \mathrm{len}\, R}, i), R) = R_i$.

3. More on the Decomposition of a Vector in a Basis

We now state a number of propositions:

(17) $v_1 + w_1 \rightarrow b_1 = (v_1 \rightarrow b_1) + (w_1 \rightarrow b_1)$.

(18) $a \cdot v_1 \to b_1 = a \cdot (v_1 \to b_1)$.

(19) If $i \in \operatorname{dom} b_1$, then $(b_1)_i \to b_1 = \operatorname{Line}(I_K^{\operatorname{len} b_1 \times \operatorname{len} b_1}, i)$.

(20) $0_{(V_1)} \to b_1 = \operatorname{len} b_1 \mapsto 0_K$.

(21) $\operatorname{len} b_1 = \dim(V_1)$.

(22)(i) $\operatorname{rng}(b_1 \restriction m)$ is a linearly independent subset of V_1, and

(ii) for every subset A of V_1 such that $A = \operatorname{rng}(b_1 \restriction m)$ holds $b_1 \restriction m$ is an ordered basis of $\operatorname{Lin}(A)$.

(23)(i) $\operatorname{rng}((b_1)_{\lfloor m})$ is a linearly independent subset of V_1, and

(ii) for every subset A of V_1 such that $A = \operatorname{rng}((b_1)_{\lfloor m})$ holds $(b_1)_{\lfloor m}$ is an ordered basis of $\operatorname{Lin}(A)$.

(24) Let W_1, W_2 be subspaces of V_1. Suppose $W_1 \cap W_2 = \mathbf{0}_{(V_1)}$. Let b_1 be an ordered basis of W_1, b_2 be an ordered basis of W_2, and b be an ordered basis of $W_1 + W_2$. Suppose $b = b_1 \frown b_2$. Let v, v_1, v_2 be vectors of $W_1 + W_2$, w_1 be a vector of W_1, and w_2 be a vector of W_2. If $v = v_1 + v_2$ and $v_1 = w_1$ and $v_2 = w_2$, then $v \to b = (w_1 \to b_1) \frown (w_2 \to b_2)$.

(25) Let W_1 be a subspace of V_1. Suppose $W_1 = \Omega_{(V_1)}$. Let w be a vector of W_1, v be a vector of V_1, and w_1 be an ordered basis of W_1. If $v = w$ and $b_1 = w_1$, then $v \to b_1 = w \to w_1$.

(26) Let W_1, W_2 be subspaces of V_1. Suppose $W_1 \cap W_2 = \mathbf{0}_{(V_1)}$. Let w_1 be an ordered basis of W_1 and w_2 be an ordered basis of W_2. Then $w_1 \frown w_2$ is an ordered basis of $W_1 + W_2$.

4. Properties of Matrices of Linear Transformations

Let us consider K, V_1, V_2, f, B_1, b_2. Then $\operatorname{AutMt}(f, B_1, b_2)$ is a matrix over K of dimension $\operatorname{len} B_1 \times \operatorname{len} b_2$.

Let S be a 1-sorted structure and let R be a binary relation. The functor $R \restriction S$ is defined as follows:

(Def. 1) $R \restriction S = R \restriction$the carrier of S.

The following proposition is true

(27) Let f be a linear transformation from V_1 to V_2, W_1, W_2 be subspaces of V_1, and U_1, U_2 be subspaces of V_2. Suppose if $\dim(W_1) = 0$, then $\dim(U_1) = 0$ and if $\dim(W_2) = 0$, then $\dim(U_2) = 0$ and V_2 is the direct sum of U_1 and U_2. Let f_1 be a linear transformation from W_1 to U_1 and f_2 be a linear transformation from W_2 to U_2. Suppose $f_1 = f \restriction W_1$ and $f_2 = f \restriction W_2$. Let w_1 be an ordered basis of W_1, w_2 be an ordered basis of W_2, u_1 be an ordered basis of U_1, and u_2 be an ordered basis of U_2. Suppose $w_1 \frown w_2 = b_1$ and $u_1 \frown u_2 = b_2$. Then $\operatorname{AutMt}(f, b_1, b_2) =$ the 0_K-block diagonal of $\langle \operatorname{AutMt}(f_1, w_1, u_1), \operatorname{AutMt}(f_2, w_2, u_2) \rangle$.

Let us consider K, V_1, V_2, let f be a function from V_1 into V_2, let B_1 be a finite sequence of elements of V_1, and let b_2 be an ordered basis of V_2. Let us assume that $\operatorname{len} B_1 = \operatorname{len} b_2$. The functor AutEqMt$(f, B_1, b_2)$ yielding a matrix over K of dimension $\operatorname{len} B_1 \times \operatorname{len} B_1$ is defined by:

(Def. 2) $\mathrm{AutEqMt}(f, B_1, b_2) = \mathrm{AutMt}(f, B_1, b_2)$.

The following propositions are true:

(28) $\mathrm{AutMt}(\mathrm{id}_{(V_1)}, b_1, b_1) = I_K^{\operatorname{len} b_1 \times \operatorname{len} b_1}$.

(29) $\mathrm{AutEqMt}(\mathrm{id}_{(V_1)}, b_1, b_1')$ is invertible and $\mathrm{AutEqMt}(\mathrm{id}_{(V_1)}, b_1', b_1) = (\mathrm{AutEqMt}(\mathrm{id}_{(V_1)}, b_1, b_1'))^{\smile}$.

(30) If $\operatorname{len} p_1 = \operatorname{len} p_2$ and $\operatorname{len} p_1 = \operatorname{len} B_1$ and $\operatorname{len} p_1 > 0$ and $j \in \operatorname{dom} b_1$ and for every i such that $i \in \operatorname{dom} p_2$ holds $p_2(i) = ((B_1)_i \rightarrow b_1)(j)$, then $p_1 \cdot p_2 = (\sum \mathrm{lmlt}(p_1, B_1) \rightarrow b_1)(j)$.

(31) If $\operatorname{len} b_1 > 0$ and f is linear, then $\mathrm{LineVec2Mx}(v_1 \rightarrow b_1) \cdot \mathrm{AutMt}(f, b_1, b_2) = \mathrm{LineVec2Mx}(f(v_1) \rightarrow b_2)$.

5. Linear Transformations of Matrices

Let us consider K, V_1, V_2, b_1, B_2 and let M be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} B_2$. The functor Mx2Tran$(M, b_1, B_2)$ yielding a function from V_1 into V_2 is defined by:

(Def. 3) For every vector v of V_1 holds $(\mathrm{Mx2Tran}(M, b_1, B_2))(v) = \sum \mathrm{lmlt}(\mathrm{Line}(\mathrm{LineVec2Mx}(v \rightarrow b_1) \cdot M, 1), B_2)$.

Next we state two propositions:

(32) For every matrix M over K of dimension $\operatorname{len} b_1 \times \operatorname{len} b_2$ such that $\operatorname{len} b_1 > 0$ holds $\mathrm{LineVec2Mx}((\mathrm{Mx2Tran}(M, b_1, b_2))(v_1) \rightarrow b_2) = \mathrm{LineVec2Mx}(v_1 \rightarrow b_1) \cdot M$.

(33) For every matrix M over K of dimension $\operatorname{len} b_1 \times \operatorname{len} B_2$ such that $\operatorname{len} b_1 = 0$ holds $(\mathrm{Mx2Tran}(M, b_1, B_2))(v_1) = 0_{(V_2)}$.

Let us consider K, V_1, V_2, b_1, B_2 and let M be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} B_2$. Then Mx2Tran$(M, b_1, B_2)$ is a linear transformation from V_1 to V_2.

Next we state three propositions:

(34) If f is linear, then $\mathrm{Mx2Tran}(\mathrm{AutMt}(f, b_1, b_2), b_1, b_2) = f$.

(35) For all matrices A, B over K such that $i \in \operatorname{dom} A$ and $\operatorname{width} A = \operatorname{len} B$ holds $\mathrm{LineVec2Mx}\,\mathrm{Line}(A, i) \cdot B = \mathrm{LineVec2Mx}\,\mathrm{Line}(A \cdot B, i)$.

(36) For every matrix M over K of dimension $\operatorname{len} b_1 \times \operatorname{len} b_2$ holds $\mathrm{AutMt}(\mathrm{Mx2Tran}(M, b_1, b_2), b_1, b_2) = M$.

Let us consider n, m, K, let A be a matrix over K of dimension $n \times m$, and let B be a matrix over K. Then $A + B$ is a matrix over K of dimension $n \times m$.

The following propositions are true:

(37) For all matrices A, B over K of dimension $\operatorname{len} b_1 \times \operatorname{len} B_2$ holds $\mathrm{Mx2Tran}(A+B, b_1, B_2) = \mathrm{Mx2Tran}(A, b_1, B_2) + \mathrm{Mx2Tran}(B, b_1, B_2)$.

(38) For every matrix A over K of dimension $\operatorname{len} b_1 \times \operatorname{len} B_2$ holds $a \cdot \mathrm{Mx2Tran}(A, b_1, B_2) = \mathrm{Mx2Tran}(a \cdot A, b_1, B_2)$.

(39) For all matrices A, B over K of dimension $\operatorname{len} b_1 \times \operatorname{len} b_2$ such that $\mathrm{Mx2Tran}(A, b_1, b_2) = \mathrm{Mx2Tran}(B, b_1, b_2)$ holds $A = B$.

(40) Let A be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} b_2$ and B be a matrix over K of dimension $\operatorname{len} b_2 \times \operatorname{len} B_3$. Suppose width $A = \operatorname{len} B$. Let A_1 be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} B_3$. If $A_1 = A \cdot B$, then $\mathrm{Mx2Tran}(A_1, b_1, B_3) = \mathrm{Mx2Tran}(B, b_2, B_3) \cdot \mathrm{Mx2Tran}(A, b_1, b_2)$.

(41) Let A be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} b_2$. Suppose $\operatorname{len} b_1 > 0$ and $\operatorname{len} b_2 > 0$. Then $v_1 \in \ker \mathrm{Mx2Tran}(A, b_1, b_2)$ if and only if $v_1 \rightarrow b_1 \in$ the space of solutions of A^{T}.

(42) V_1 is trivial iff $\dim(V_1) = 0$.

(43) Let V_1, V_2 be vector spaces over K and f be a linear transformation from V_1 to V_2. Then f is one-to-one if and only if $\ker f = \mathbf{0}_{(V_1)}$.

Let us consider K and let V_1 be a vector space over K. Then $\mathrm{id}_{(V_1)}$ is a linear transformation from V_1 to V_1.

Let us consider K, let V_1, V_2 be vector spaces over K, and let f, g be linear transformations from V_1 to V_2. Then $f+g$ is a linear transformation from V_1 to V_2.

Let us consider K, let V_1, V_2 be vector spaces over K, let f be a linear transformation from V_1 to V_2, and let us consider a. Then $a \cdot f$ is a linear transformation from V_1 to V_2.

Let us consider K, let V_1, V_2, V_3 be vector spaces over K, let f_3 be a linear transformation from V_1 to V_2, and let f_4 be a linear transformation from V_2 to V_3. Then $f_4 \cdot f_3$ is a linear transformation from V_1 to V_3.

One can prove the following propositions:

(44) For every matrix A over K of dimension $\operatorname{len} b_1 \times \operatorname{len} b_2$ such that $\mathrm{rk}(A) = \operatorname{len} b_1$ holds $\mathrm{Mx2Tran}(A, b_1, b_2)$ is one-to-one.

(45) $\mathrm{MX2FinS}(I_K^{n \times n})$ is an ordered basis of the n-dimension vector space over K.

(46) Let M be an ordered basis of the $\operatorname{len} b_2$-dimension vector space over K. Suppose $M = \mathrm{MX2FinS}(I_K^{\operatorname{len} b_2 \times \operatorname{len} b_2})$. Let v_1 be a vector of the $\operatorname{len} b_2$-dimension vector space over K. Then $v_1 \rightarrow M = v_1$.

(47) Let M be an ordered basis of the $\operatorname{len} b_2$-dimension vector space over K. Suppose $M = \mathrm{MX2FinS}(I_K^{\operatorname{len} b_2 \times \operatorname{len} b_2})$. Let A be a matrix over K of dimension $\operatorname{len} b_1 \times \operatorname{len} M$. If $A = \mathrm{AutMt}(f, b_1, b_2)$ and f is linear, then $(\mathrm{Mx2Tran}(A, b_1, M))(v_1) = f(v_1) \rightarrow b_2$.

Let K be an add-associative right zeroed right complementable Abelian associative well unital distributive non empty double loop structure, let V_1, V_2 be Abelian add-associative right zeroed right complementable vector space-like non empty vector space structures over K, let W be a subspace of V_1, and let f be a function from V_1 into V_2. Then $f \restriction W$ is a function from W into V_2.

Let K be a field, let V_1, V_2 be vector spaces over K, let W be a subspace of V_1, and let f be a linear transformation from V_1 to V_2. Then $f \restriction W$ is a linear transformation from W to V_2.

6. The Main Theorems

The following propositions are true:

(48) For every linear transformation f from V_1 to V_2 holds $\operatorname{rank} f = \operatorname{rk}(\operatorname{AutMt}(f, b_1, b_2))$.

(49) For every matrix M over K of dimension $\operatorname{len} b_1 \times \operatorname{len} b_2$ holds $\operatorname{rank} \operatorname{Mx2Tran}(M, b_1, b_2) = \operatorname{rk}(M)$.

(50) For every linear transformation f from V_1 to V_2 such that $\dim(V_1) = \dim(V_2)$ holds $\ker f$ is non trivial iff $\operatorname{Det} \operatorname{AutEqMt}(f, b_1, b_2) = 0_K$.

(51) Let f be a linear transformation from V_1 to V_2 and g be a linear transformation from V_2 to V_3. If $g \restriction \operatorname{im} f$ is one-to-one, then $\operatorname{rank}(g \cdot f) = \operatorname{rank} f$ and $\operatorname{nullity}(g \cdot f) = \operatorname{nullity} f$.

References

[1] Jesse Alama. The rank+nullity theorem. *Formalized Mathematics*, 15(**3**):137–142, 2007.
[2] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[3] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Czesław Byliński. Binary operations applied to finite sequences. *Formalized Mathematics*, 1(**4**):643–649, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[10] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.
[11] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Formalized Mathematics*, 3(**2**):275–278, 1992.
[12] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[13] Robert Milewski. Associated matrix of linear map. *Formalized Mathematics*, 5(**3**):339–345, 1996.
[14] Michał Muzalewski. Rings and modules – part II. *Formalized Mathematics*, 2(**4**):579–585, 1991.

[15] Karol Pąk. Basic properties of the rank of matrices over a field. *Formalized Mathematics*, 15(**4**):199–211, 2007.
[16] Karol Pąk. Block diagonal matrices. *Formalized Mathematics*, 16(**3**):259–267, 2008.
[17] Karol Pąk. Solutions of linear equations. *Formalized Mathematics*, 16(**1**):81–90, 2008.
[18] Wojciech A. Trybulec. Basis of vector space. *Formalized Mathematics*, 1(**5**):883–885, 1990.
[19] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[20] Wojciech A. Trybulec. Linear combinations in vector space. *Formalized Mathematics*, 1(**5**):877–882, 1990.
[21] Wojciech A. Trybulec. Operations on subspaces in vector space. *Formalized Mathematics*, 1(**5**):871–876, 1990.
[22] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Formalized Mathematics*, 1(**5**):865–870, 1990.
[23] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[24] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[25] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[26] Xiaopeng Yue, Xiquan Liang, and Zhongpin Sun. Some properties of some special matrices. *Formalized Mathematics*, 13(**4**):541–547, 2005.
[27] Katarzyna Zawadzka. The sum and product of finite sequences of elements of a field. *Formalized Mathematics*, 3(**2**):205–211, 1992.
[28] Katarzyna Zawadzka. The product and the determinant of matrices with entries in a field. *Formalized Mathematics*, 4(**1**):1–8, 1993.
[29] Mariusz Żynel. The Steinitz theorem and the dimension of a vector space. *Formalized Mathematics*, 5(**3**):423–428, 1996.

Received May 13, 2008