FORMALIZED MATHEMATICS
Vol. **16**, No. **1**, Pages 51–55, **2008**

The First Mean Value Theorem for Integrals

Keiko Narita
Hirosaki-city
Aomori, Japan

Noboru Endou
Gifu National College of Technology
Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article, we prove the first mean value theorem for integrals [16]. The formalization of various theorems about the properties of the Lebesgue integral is also presented.

MML identifier: `MESFUNC7`, version: `7.8.09 4.97.1001`

The notation and terminology used in this paper are introduced in the following articles: [20], [2], [17], [6], [1], [4], [21], [22], [11], [3], [9], [8], [10], [18], [19], [5], [13], [12], [14], [15], and [7].

1. Lemmas for Extended Real Valued Functions

For simplicity, we use the following convention: X is a non empty set, S is a σ-field of subsets of X, M is a σ-measure on S, f, g are partial functions from X to $\overline{\mathbb{R}}$, and E is an element of S.

One can prove the following three propositions:

(1) If for every element x of X such that $x \in \operatorname{dom} f$ holds $f(x) \leq g(x)$, then $g - f$ is non-negative.

(2) For every set Y and for every partial function f from X to $\overline{\mathbb{R}}$ and for every real number r holds $(r\, f){\restriction}Y = r\,(f{\restriction}Y)$.

(3) Suppose f is integrable on M and g is integrable on M and $g - f$ is non-negative. Then there exists an element E of S such that $E = \operatorname{dom} f \cap \operatorname{dom} g$ and $\int f{\restriction}E \,\mathrm{d}M \leq \int g{\restriction}E \,\mathrm{d}M$.

ISSN 1426–2630(p), 1898-9934(e)

2. σ-Finite Sets

Let us consider X. One can verify that there exists a partial function from X to $\overline{\mathbb{R}}$ which is non-negative.

Let us consider X, f. Then $|f|$ is a non-negative partial function from X to $\overline{\mathbb{R}}$.

Next we state the proposition

(4) Suppose f is integrable on M. Then there exists a function F from $\mathbb{N}$ into S such that

(i) for every element n of $\mathbb{N}$ holds $F(n) = \operatorname{dom} f \cap \text{GTE-dom}(|f|, \overline{\mathbb{R}}(\frac{1}{n+1}))$,

(ii) $\operatorname{dom} f \setminus \text{EQ-dom}(f, 0_{\overline{\mathbb{R}}}) = \bigcup \operatorname{rng} F$, and

(iii) for every element n of $\mathbb{N}$ holds $F(n) \in S$ and $M(F(n)) < +\infty$.

3. The First Mean Value Theorem for Integrals

Let F be a binary relation. We introduce F is extreal-yielding as a synonym of F is extended real-valued.

Let k be a natural number and let x be an element of $\overline{\mathbb{R}}$. Then $k \mapsto x$ is a finite sequence of elements of $\overline{\mathbb{R}}$.

Let us note that there exists a finite sequence which is extreal-yielding.

The binary operation $\cdot_{\overline{\mathbb{R}}}$ on $\overline{\mathbb{R}}$ is defined by:

(Def. 2)[1] For all elements x, y of $\overline{\mathbb{R}}$ holds $\cdot_{\overline{\mathbb{R}}}(x, y) = x \cdot y$.

One can check that $\cdot_{\overline{\mathbb{R}}}$ is commutative and associative.

One can prove the following proposition

(5) $\mathbf{1}_{\cdot_{\overline{\mathbb{R}}}} = 1$.

One can check that $\cdot_{\overline{\mathbb{R}}}$ is unital.

Let F be an extreal-yielding finite sequence. The functor $\prod F$ yields an element of $\overline{\mathbb{R}}$ and is defined by:

(Def. 3) There exists a finite sequence f of elements of $\overline{\mathbb{R}}$ such that $f = F$ and $\prod F = \cdot_{\overline{\mathbb{R}}} \circledast f$.

Let x be an element of $\overline{\mathbb{R}}$ and let n be a natural number. Note that $n \mapsto x$ is extreal-yielding.

Let x be an element of $\overline{\mathbb{R}}$ and let k be a natural number. The functor x^k is defined by:

(Def. 4) $x^k = \prod(k \mapsto x)$.

Let x be an element of $\overline{\mathbb{R}}$ and let k be a natural number. Then x^k is an extended real number.

Let us note that $\varepsilon_{\overline{\mathbb{R}}}$ is extreal-yielding.

[1]The definition (Def. 1) has been removed.

Let r be an element of $\overline{\mathbb{R}}$. Note that $\langle r\rangle$ is extreal-yielding.

We now state two propositions:

(6) $\prod(\varepsilon_{\overline{\mathbb{R}}}) = 1$.

(7) For every element r of $\overline{\mathbb{R}}$ holds $\prod\langle r\rangle = r$.

Let f, g be extreal-yielding finite sequences. Observe that $f \frown g$ is extreal-yielding.

We now state three propositions:

(8) For every extreal-yielding finite sequence F and for every element r of $\overline{\mathbb{R}}$ holds $\prod(F \frown \langle r\rangle) = \prod F \cdot r$.

(9) For every element x of $\overline{\mathbb{R}}$ holds $x^1 = x$.

(10) For every element x of $\overline{\mathbb{R}}$ and for every natural number k holds $x^{k+1} = x^k \cdot x$.

Let k be a natural number and let us consider X, f. The functor f^k yields a partial function from X to $\overline{\mathbb{R}}$ and is defined by:

(Def. 5) $\mathrm{dom}(f^k) = \mathrm{dom}\, f$ and for every element x of X such that $x \in \mathrm{dom}(f^k)$ holds $f^k(x) = f(x)^k$.

Next we state several propositions:

(11) For every element x of $\overline{\mathbb{R}}$ and for every real number y and for every natural number k such that $x = y$ holds $x^k = y^k$.

(12) For every element x of $\overline{\mathbb{R}}$ and for every natural number k such that $0 \leq x$ holds $0 \leq x^k$.

(13) For every natural number k such that $1 \leq k$ holds $+\infty^k = +\infty$.

(14) Let k be a natural number and given X, S, f, E. If $E \subseteq \mathrm{dom}\, f$ and f is measurable on E, then $|f|^k$ is measurable on E.

(15) Suppose $\mathrm{dom}\, f \cap \mathrm{dom}\, g = E$ and f is finite and g is finite and f is measurable on E and g is measurable on E. Then $f\, g$ is measurable on E.

(16) If $\mathrm{rng}\, f$ is bounded, then f is finite.

(17) Let M be a σ-measure on S, f, g be partial functions from X to $\overline{\mathbb{R}}$, E be an element of S, and F be a non empty subset of $\overline{\mathbb{R}}$. Suppose $\mathrm{dom}\, f \cap \mathrm{dom}\, g = E$ and $\mathrm{rng}\, f = F$ and g is finite and f is measurable on E and $\mathrm{rng}\, f$ is bounded and g is integrable on M. Then $(f\, g)\restriction E$ is integrable on M and there exists an element c of $\mathbb{R}$ such that $c \geq \inf F$ and $c \leq \sup F$ and $\int (f\,|g|)\restriction E\, \mathrm{d}M = \overline{\mathbb{R}}(c) \cdot \int |g|\restriction E\, \mathrm{d}M$.

4. Selected Properties of Integrals

We use the following convention: E_1, E_2 denote elements of S, x, A denote sets, and a, b denote real numbers.

The following propositions are true:

(18) $|f|\upharpoonright A = |f\upharpoonright A|$.

(19) $\operatorname{dom}(|f| + |g|) = \operatorname{dom} f \cap \operatorname{dom} g$ and $\operatorname{dom}|f+g| \subseteq \operatorname{dom}|f|$.

(20) $|f|\upharpoonright \operatorname{dom}|f+g| + |g|\upharpoonright \operatorname{dom}|f+g| = (|f| + |g|)\upharpoonright \operatorname{dom}|f+g|$.

(21) If $x \in \operatorname{dom}|f+g|$, then $|f+g|(x) \leq (|f| + |g|)(x)$.

(22) Suppose f is integrable on M and g is integrable on M. Then there exists an element E of S such that $E = \operatorname{dom}(f+g)$ and $\int |f+g|\upharpoonright E \, \mathrm{d}M \leq \int |f|\upharpoonright E \, \mathrm{d}M + \int |g|\upharpoonright E \, \mathrm{d}M$.

(23) $\max_{+}(\chi_{A,X}) = \chi_{A,X}$.

(24) If $M(E) < +\infty$, then $\chi_{E,X}$ is integrable on M and $\int \chi_{E,X} \, \mathrm{d}M = M(E)$ and $\int \chi_{E,X}\upharpoonright E \, \mathrm{d}M = M(E)$.

(25) If $M(E_1 \cap E_2) < +\infty$, then $\int \chi_{(E_1),X}\upharpoonright E_2 \, \mathrm{d}M = M(E_1 \cap E_2)$.

(26) Suppose f is integrable on M and $E \subseteq \operatorname{dom} f$ and $M(E) < +\infty$ and for every element x of X such that $x \in E$ holds $a \leq f(x) \leq b$. Then $\overline{\mathbb{R}}(a) \cdot M(E) \leq \int f\upharpoonright E \, \mathrm{d}M \leq \overline{\mathbb{R}}(b) \cdot M(E)$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):163–171, 1991.

[5] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):173–183, 1991.

[6] Józef Białas. The σ-additive measure theory. *Formalized Mathematics*, 2(**2**):263–270, 1991.

[7] Józef Białas. Some properties of the intervals. *Formalized Mathematics*, 5(**1**):21–26, 1996.

[8] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[9] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[10] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[11] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[12] Noboru Endou and Yasunari Shidama. Integral of measurable function. *Formalized Mathematics*, 14(**2**):53–70, 2006.

[13] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Basic properties of extended real numbers. *Formalized Mathematics*, 9(**3**):491–494, 2001.

[14] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definitions and basic properties of measurable functions. *Formalized Mathematics*, 9(**3**):495–500, 2001.

[15] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. The measurability of extended real valued functions. *Formalized Mathematics*, 9(**3**):525–529, 2001.

[16] P. R. Halmos. *Measure Theory*. Springer-Verlag, 1987.

[17] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[18] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.

[19] Andrzej Nędzusiak. σ-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.

[20] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[21] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[22] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received October 30, 2007