String Rewriting Systems

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Summary. Basing on the definitions from [15], semi-Thue systems, Thue systems, and direct derivations are introduced. Next, the standard reduction relation is defined that, in turn, is used to introduce derivations using the theory from [1]. Finally, languages generated by rewriting systems are defined as all strings reachable from an initial word. This is followed by the introduction of the equivalence of semi-Thue systems with respect to the initial word.

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The notation and terminology used here are introduced in the following papers: [11], [13], [8], [16], [10], [4], [17], [14], [7], [18], [2], [1], [3], [12], [5], [6], and [9].

1. Preliminaries

We adopt the following convention: x denotes a set, k, l denote natural numbers, and p, q denote finite sequences.

Next we state two propositions:

- (1) If $k \notin \text{dom } p \text{ and } k+1 \in \text{dom } p$, then k=0.
- (2) If k > len p and $k \le \text{len}(p \cap q)$, then there exists l such that k = len p + l and $l \ge 1$ and $l \le \text{len } q$.

In the sequel R denotes a binary relation and $p,\,q$ denote reduction sequences w.r.t. R.

Next we state two propositions:

- (3) If $k \ge 1$, then $p \upharpoonright k$ is a reduction sequence w.r.t. R.
- (4) If $k \in \text{dom } p$, then there exists q such that len q = k and q(1) = p(1) and q(len q) = p(k).

2. Finite 0-sequence Yielding Functions and Finite Sequences

Let f be a function. We say that f is finite-0-sequence-yielding if and only if:

(Def. 1) If $x \in \text{dom } f$, then f(x) is a finite 0-sequence.

Let us mention that \emptyset is finite-0-sequence-yielding.

Let f be a finite 0-sequence. Observe that $\langle f \rangle$ is finite-0-sequence-yielding. Let us observe that there exists a function which is finite-0-sequence-yielding.

Let p be a finite-0-sequence-yielding function and let us consider x. Then p(x) is a finite 0-sequence.

One can verify that there exists a finite sequence which is finite-0-sequence-yielding.

Let E be a set. Note that every finite sequence of elements of E^{ω} is finite-0-sequence-yielding.

Let p, q be finite-0-sequence-yielding finite sequences. Observe that $p \cap q$ is finite-0-sequence-yielding.

3. Concatenation of a Finite 0-sequence with All Elements of a Finite 0-sequence Yielding Function

Let s be a finite 0-sequence and let p be a finite-0-sequence-yielding function. The functor s + p yields a finite-0-sequence-yielding function and is defined by:

(Def. 2) dom(s+p) = dom p and for every x such that $x \in dom p$ holds $(s+p)(x) = s \cap p(x)$.

The functor p + s yielding a finite-0-sequence-yielding function is defined by:

(Def. 3) dom(p+s) = dom p and for every x such that $x \in dom p$ holds $(p+s)(x) = p(x) \cap s$.

Let s be a finite 0-sequence and let p be a finite-0-sequence-yielding finite sequence. Note that s + p is finite sequence-like and p + s is finite sequence-like.

We adopt the following convention: E denotes a set, s, t denote finite 0-sequences, and p, q denote finite-0-sequence-yielding finite sequences.

The following propositions are true:

- (5) $\operatorname{len}(s+p) = \operatorname{len} p$ and $\operatorname{len}(p+s) = \operatorname{len} p$.
- (6) $\langle \rangle_E + p = p \text{ and } p + \langle \rangle_E = p.$
- (7) $s + (t+p) = s \land t + p \text{ and } p + t + s = p + t \land s.$
- (8) s + (p+t) = (s+p) + t.
- (9) $s + p \cap q = (s + p) \cap (s + q)$ and $p \cap q + s = (p + s) \cap (q + s)$.

4. Semi-Thue Systems and Thue Systems

Let E be a set, let p be a finite sequence of elements of E^{ω} , and let k be a natural number. Then p(k) is an element of E^{ω} .

Let E be a set, let k be a natural number, and let s be an element of E^{ω} . Then $k \mapsto s$ is a finite sequence of elements of E^{ω} .

Let E be a set, let s be an element of E^{ω} , and let p be a finite sequence of elements of E^{ω} . Then s+p is a finite sequence of elements of E^{ω} . Then p+s is a finite sequence of elements of E^{ω} .

Let E be a set. A semi-Thue-system of E is a binary relation on E^{ω} .

In the sequel E is a set and S, T, U are semi-Thue-systems of E.

Let S be a binary relation. Observe that $S \cup S^{\sim}$ is symmetric.

Let us consider E. One can check that there exists a semi-Thue-system of E which is symmetric.

Let E be a set. A Thue-system of E is a symmetric semi-Thue-system of E.

5. Direct Derivations

We follow the rules: s, t, s_1, t_1, u, v, w are elements of E^{ω} and p is a finite sequence of elements of E^{ω} .

Let us consider E, S, s, t. The predicate $s \to_S t$ is defined by:

(Def. 4) $\langle s, t \rangle \in S$.

Let us consider E, S, s, t. The predicate $s \Rightarrow_S t$ is defined as follows:

(Def. 5) There exist v, w, s_1 , t_1 such that $s = v \cap s_1 \cap w$ and $t = v \cap t_1 \cap w$ and $s_1 \to_S t_1$.

The following propositions are true:

- (10) If $s \to_S t$, then $s \Rightarrow_S t$.
- (11) If $s \Rightarrow_S s$, then there exist v, w, s_1 such that $s = v \land s_1 \land w$ and $s_1 \rightarrow_S s_1$.
- (12) If $s \Rightarrow_S t$, then $u \cap s \Rightarrow_S u \cap t$ and $s \cap u \Rightarrow_S t \cap u$.
- (13) If $s \Rightarrow_S t$, then $u \cap s \cap v \Rightarrow_S u \cap t \cap v$.
- (14) If $s \to_S t$, then $u \cap s \Rightarrow_S u \cap t$ and $s \cap u \Rightarrow_S t \cap u$.
- (15) If $s \to_S t$, then $u \cap s \cap v \Rightarrow_S u \cap t \cap v$.
- (16) If S is a Thue-system of E and $s \to_S t$, then $t \to_S s$.
- (17) If S is a Thue-system of E and $s \Rightarrow_S t$, then $t \Rightarrow_S s$.
- (18) If $S \subseteq T$ and $s \to_S t$, then $s \to_T t$.
- (19) If $S \subseteq T$ and $s \Rightarrow_S t$, then $s \Rightarrow_T t$.
- (20) $s \not\Rightarrow_{\emptyset_{E^{\omega},E^{\omega}}} t$.
- (21) If $s \Rightarrow_{S \cup T} t$, then $s \Rightarrow_S t$ or $s \Rightarrow_T t$.

6. REDUCTION RELATION

Let us consider E. Then id_E is a binary relation on E.

Let us consider E, S. The functor \Rightarrow_S yielding a binary relation on E^{ω} is defined as follows:

(Def. 6) $\langle s, t \rangle \in \Rightarrow_S \text{ iff } s \Rightarrow_S t.$

The following propositions are true:

- (22) $S \subseteq \Rightarrow_S$.
- (23) Suppose p is a reduction sequence w.r.t. \Rightarrow_S . Then p + u is a reduction sequence w.r.t. \Rightarrow_S and u + p is a reduction sequence w.r.t. \Rightarrow_S .
- (24) If p is a reduction sequence w.r.t. \Rightarrow_S , then (t+p)+u is a reduction sequence w.r.t. \Rightarrow_S .
- (25) If S is a Thue-system of E, then $\Rightarrow_S = (\Rightarrow_S)^{\smile}$.
- (26) If $S \subseteq T$, then $\Rightarrow_S \subseteq \Rightarrow_T$.
- $(27) \quad \Rightarrow_{\mathrm{id}_{E^{\omega}}} = \mathrm{id}_{E^{\omega}}.$
- $(28) \quad \Rightarrow_{S \cup \mathrm{id}_{F^{\omega}}} = \Rightarrow_{S} \cup \mathrm{id}_{E^{\omega}}.$
- $(29) \Rightarrow_{\emptyset_{E^{\omega}}} = \emptyset_{E^{\omega}, E^{\omega}}.$
- (30) If $s \Rightarrow_S t$, then $s \Rightarrow_S t$.
- $(31) \Rightarrow_{\Rightarrow_S} = \Rightarrow_S.$

7. Derivations

Let us consider E, S, s, t. The predicate $s \Rightarrow_S^* t$ is defined by:

(Def. 7) \Rightarrow_S reduces s to t.

One can prove the following propositions:

- (32) $s \Rightarrow_S^* s$.
- (33) If $s \Rightarrow_S t$, then $s \Rightarrow_S^* t$.
- (34) If $s \to_S t$, then $s \Rightarrow_S^* t$.
- (35) If $s \Rightarrow_S^* t$ and $t \Rightarrow_S^* u$, then $s \Rightarrow_S^* u$.
- (36) If $s \Rightarrow_S^* t$, then $s \cap u \Rightarrow_S^* t \cap u$ and $u \cap s \Rightarrow_S^* u \cap t$.
- (37) If $s \Rightarrow_S^* t$, then $u \cap s \cap v \Rightarrow_S^* u \cap t \cap v$.
- (38) If $s \Rightarrow_S^* t$ and $u \Rightarrow_S^* v$, then $s \cap u \Rightarrow_S^* t \cap v$ and $u \cap s \Rightarrow_S^* v \cap t$.
- (39) If S is a Thue-system of E and $s \Rightarrow_S^* t$, then $t \Rightarrow_S^* s$.
- (40) If $S \subseteq T$ and $s \Rightarrow_S^* t$, then $s \Rightarrow_T^* t$.
- $(41) \quad s \Rightarrow_S^* t \text{ iff } s \Rightarrow_{S \cup \mathrm{id}_{E^\omega}}^* t.$
- (42) If $s \Rightarrow_{\emptyset_{E^{\omega}}}^* t$, then s = t.
- (43) If $s \Rightarrow_{\Rightarrow_S}^* t$, then $s \Rightarrow_S^* t$.

- (44) If $s \Rightarrow_S^* t$ and $u \Rightarrow_{\{\langle s, t \rangle\}} v$, then $u \Rightarrow_S^* v$.
- (45) If $s \Rightarrow_S^* t$ and $u \Rightarrow_{S \cup \{\langle s, t \rangle\}}^* v$, then $u \Rightarrow_S^* v$.

8. Languages Generated by Semi-Thue Systems

Let us consider E, S, w. The functor $\mathrm{Lang}(w,S)$ yields a subset of E^{ω} and is defined by:

(Def. 8) $\operatorname{Lang}(w, S) = \{s : w \Rightarrow_S^* s\}.$

Next we state two propositions:

- (46) $s \in \text{Lang}(w, S) \text{ iff } w \Rightarrow_S^* s.$
- (47) $w \in \text{Lang}(w, S)$.

Let E be a non empty set, let S be a semi-Thue-system of E, and let w be an element of E^{ω} . Note that Lang(w, S) is non empty.

We now state four propositions:

- (48) If $S \subseteq T$, then $\text{Lang}(w, S) \subseteq \text{Lang}(w, T)$.
- (49) $\operatorname{Lang}(w, S) = \operatorname{Lang}(w, S \cup \operatorname{id}_{E^{\omega}}).$
- (50) Lang $(w, \emptyset_{E^{\omega}, E^{\omega}}) = \{w\}.$
- (51) $\operatorname{Lang}(w, \operatorname{id}_{E^{\omega}}) = \{w\}.$

9. Equivalence of Semi-Thue Systems

Let us consider E, S, T, w. We say that S and T are equivalent wrt w if and only if:

(Def. 9) Lang(w, S) = Lang(w, T).

The following propositions are true:

- (52) S and S are equivalent wrt w.
- (53) If S and T are equivalent wrt w, then T and S are equivalent wrt w.
- (54) Suppose S and T are equivalent wrt w and T and U are equivalent wrt w. Then S and U are equivalent wrt w.
- (55) S and $S \cup \mathrm{id}_{E^{\omega}}$ are equivalent wrt w.
- (56) Suppose S and T are equivalent wrt w and $S \subseteq U$ and $U \subseteq T$. Then S and U are equivalent wrt w and U and T are equivalent wrt w.
- (57) S and \Rightarrow_S are equivalent wrt w.
- (58) If S and T are equivalent wrt w and $\Rightarrow_{S \cup T}$ reduces w to s, then \Rightarrow_S reduces w to s.
- (59) If S and T are equivalent wrt w and $w \Rightarrow_{S \cup T}^* s$, then $w \Rightarrow_S^* s$.
- (60) If S and T are equivalent wrt w, then S and $S \cup T$ are equivalent wrt w.

- (61) If $s \Rightarrow_S t$, then S and $S \cup \{\langle s, t \rangle\}$ are equivalent wrt w.
- (62) If $s \Rightarrow_S^* t$, then S and $S \cup \{\langle s, t \rangle\}$ are equivalent wrt w.

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