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Basic Properties of Determinants of Square Matrices over a Field[1]

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Summary. In this paper I present basic properties of the determinant of square matrices over a field and selected properties of the sign of a permutation. First, I define the sign of a permutation by the requirement

$$\mathrm{sgn}(p) = \prod_{1 \leq i < j \leq n} \mathrm{sgn}(p(j) - p(i)),$$

where p is any fixed permutation of a set with n elements. I prove that the sign of a product of two permutations is the same as the product of their signs and show the relation between signs and parity of permutations. Then I consider the determinant of a linear combination of lines, the determinant of a matrix with permutated lines and the determinant of a matrix with a repeated line. Finally, at the end I prove that the determinant of a product of two square matrices is equal to the product of their determinants.

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The articles [21], [12], [27], [18], [13], [28], [7], [10], [8], [3], [4], [19], [25], [24], [16], [20], [11], [6], [5], [14], [22], [15], [31], [23], [26], [32], [1], [29], [9], [2], [17], and [30] provide the terminology and notation for this paper.

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1. The Sign of a Permutation

For simplicity, we use the following convention: x, X denote sets, i, j, k, l, n, m denote natural numbers, D denotes a non empty set, K denotes a field, a, b denote elements of K, p_1, p, q denote elements of the permutations of n-element set, P_1, P denote permutations of $\mathrm{Seg}\, n$, F denotes a function from $\mathrm{Seg}\, n$ into $\mathrm{Seg}\, n$, p_2, p_3, q_2, p_4 denote elements of the permutations of $(n+2)$-element set, and P_2 denotes a permutation of $\mathrm{Seg}(n+2)$.

Let X be a set. We introduce $\mathrm{2Set}\, X$ as a synonym of $\mathrm{TwoElementSets}(X)$.

The following three propositions are true:

(1) $X \in \mathrm{2Set}\,\mathrm{Seg}\, n$ iff there exist i, j such that $i \in \mathrm{Seg}\, n$ and $j \in \mathrm{Seg}\, n$ and $i < j$ and $X = \{i, j\}$.

(2) $\mathrm{2Set}\,\mathrm{Seg}\, 0 = \emptyset$ and $\mathrm{2Set}\,\mathrm{Seg}\, 1 = \emptyset$.

(3) For every n such that $n \geq 2$ holds $\{1, 2\} \in \mathrm{2Set}\,\mathrm{Seg}\, n$.

Let us consider n. Observe that $\mathrm{2Set}\,\mathrm{Seg}(n+2)$ is non empty and finite.

Let us consider n, x and let p_1 be an element of the permutations of n-element set. Note that $p_1(x)$ is natural.

Let us consider K. One can verify that the multiplication of K is unital and the multiplication of K is associative.

Let us consider n, K and let p_2 be an element of the permutations of $(n+2)$-element set. The functor $\text{Part-sgn}(p_2, K)$ yielding a function from $\mathrm{2Set}\,\mathrm{Seg}(n+2)$ into the carrier of K is defined by the condition (Def. 1).

(Def. 1) Let i, j be elements of $\mathbb{N}$ such that $i \in \mathrm{Seg}(n+2)$ and $j \in \mathrm{Seg}(n+2)$ and $i < j$. Then

(i) if $p_2(i) < p_2(j)$, then $(\text{Part-sgn}(p_2, K))(\{i, j\}) = \mathbf{1}_K$, and

(ii) if $p_2(i) > p_2(j)$, then $(\text{Part-sgn}(p_2, K))(\{i, j\}) = -\mathbf{1}_K$.

One can prove the following proposition

(4) Let X be an element of $\mathrm{Fin}\,\mathrm{2Set}\,\mathrm{Seg}(n+2)$. Suppose that for every x such that $x \in X$ holds $(\text{Part-sgn}(p_3, K))(x) = \mathbf{1}_K$. Then (the multiplication of K)-$\sum_X \text{Part-sgn}(p_3, K) = \mathbf{1}_K$.

In the sequel s denotes an element of $\mathrm{2Set}\,\mathrm{Seg}(n+2)$.

The following propositions are true:

(5) $(\text{Part-sgn}(p_3, K))(s) = \mathbf{1}_K$ or $(\text{Part-sgn}(p_3, K))(s) = -\mathbf{1}_K$.

(6) For all i, j such that $i \in \mathrm{Seg}(n+2)$ and $j \in \mathrm{Seg}(n+2)$ and $i < j$ and $p_3(i) = q_2(i)$ and $p_3(j) = q_2(j)$ holds $(\text{Part-sgn}(p_3, K))(\{i, j\}) = (\text{Part-sgn}(q_2, K))(\{i, j\})$.

(7) Let X be an element of $\mathrm{Fin}\,\mathrm{2Set}\,\mathrm{Seg}(n+2)$, given p_3, q_2, and F be a finite set such that $F = \{s : s \in X \wedge (\text{Part-sgn}(p_3, K))(s) \neq (\text{Part-sgn}(q_2, K))(s)\}$. Then

(i) if $\operatorname{card} F \bmod 2 = 0$, then (the multiplication of K)- $\sum_X$ Part-sgn(p_3, K) = (the multiplication of K)- $\sum_X$ Part-sgn(q_2, K), and

(ii) if $\operatorname{card} F \bmod 2 = 1$, then (the multiplication of K)- $\sum_X$ Part-sgn(p_3, K) = $-$((the multiplication of K)- $\sum_X$ Part-sgn(q_2, K)).

(8) Let P be a permutation of $\operatorname{Seg} n$. Suppose P is a transposition. Let given i, j. Suppose $i < j$. Then $P(i) = j$ if and only if the following conditions are satisfied:

(i) $i \in \operatorname{dom} P$,

(ii) $j \in \operatorname{dom} P$,

(iii) $P(i) = j$,

(iv) $P(j) = i$, and

(v) for every k such that $k \neq i$ and $k \neq j$ and $k \in \operatorname{dom} P$ holds $P(k) = k$.

(9) Let given p_3, q_2, p_4, i, j. Suppose $p_4 = p_3 \cdot q_2$ and q_2 is a transposition and $q_2(i) = j$ and $i < j$. Let given s. If (Part-sgn(p_3, K))$(s) \neq$ (Part-sgn(p_4, K))(s), then $i \in s$ or $j \in s$.

(10) Let given p_3, q_2, p_4, i, j, K. Suppose $p_4 = p_3 \cdot q_2$ and q_2 is a transposition and $q_2(i) = j$ and $i < j$ and $\mathbf{1}_K \neq -\mathbf{1}_K$. Then

(i) (Part-sgn(p_3, K))$(\{i, j\}) \neq$ (Part-sgn(p_4, K))$(\{i, j\})$, and

(ii) for every k such that $k \in \operatorname{Seg}(n+2)$ and $i \neq k$ and $j \neq k$ holds (Part-sgn(p_3, K))$(\{i, k\}) \neq$ (Part-sgn(p_4, K))$(\{i, k\})$ iff (Part-sgn(p_3, K))$(\{j, k\}) \neq$ (Part-sgn(p_4, K))$(\{j, k\})$.

Let us consider n, K and let p_2 be an element of the permutations of $(n+2)$-element set. The functor $\operatorname{sgn}(p_2, K)$ yielding an element of K is defined by:

(Def. 2) $\operatorname{sgn}(p_2, K)$ = (the multiplication of K)- $\sum_{\Omega^{\mathrm{f}}_{\mathrm{2Set\,Seg}(n+2)}}$ Part-sgn(p_2, K).

The following propositions are true:

(11) $\operatorname{sgn}(p_3, K) = \mathbf{1}_K$ or $\operatorname{sgn}(p_3, K) = -\mathbf{1}_K$.

(12) For every element I_1 of the permutations of $(n+2)$-element set such that $I_1 = \operatorname{idseq}(n+2)$ holds $\operatorname{sgn}(I_1, K) = \mathbf{1}_K$.

(13) For all p_3, q_2, p_4 such that $p_4 = p_3 \cdot q_2$ and q_2 is a transposition holds $\operatorname{sgn}(p_4, K) = -\operatorname{sgn}(p_3, K)$.

(14) For every element t_1 of the permutations of $(n+2)$-element set such that t_1 is a transposition holds $\operatorname{sgn}(t_1, K) = -\mathbf{1}_K$.

(15) Let P be a finite sequence of elements of A_{n+2} and p_3 be an element of the permutations of $(n+2)$-element set such that $p_3 = \prod P$ and for every i such that $i \in \operatorname{dom} P$ there exists an element t_2 of the permutations of $(n+2)$-element set such that $P(i) = t_2$ and t_2 is a transposition. Then

(i) if $\operatorname{len} P \bmod 2 = 0$, then $\operatorname{sgn}(p_3, K) = \mathbf{1}_K$, and

(ii) if $\operatorname{len} P \bmod 2 = 1$, then $\operatorname{sgn}(p_3, K) = -\mathbf{1}_K$.

(16) Let given i, j, n. Suppose $i < j$ and $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$. Then there exists an element t_1 of the permutations of n-element set such that t_1 is a

transposition and $t_1(i) = j$.

(17) Let p be an element of the permutations of $(k+1)$-element set. Suppose $p(k+1) \neq k+1$. Then there exists an element t_1 of the permutations of $(k+1)$-element set such that t_1 is a transposition and $t_1(p(k+1)) = k+1$ and $(t_1 \cdot p)(k+1) = k+1$.

(18) Let given X, x. Suppose $x \notin X$. Let p_5 be a permutation of $X \cup \{x\}$. If $p_5(x) = x$, then there exists a permutation p of X such that $p_5 \restriction X = p$.

(19) Let p, q be permutations of X and p_5, q_1 be permutations of $X \cup \{x\}$. If $p_5 \restriction X = p$ and $q_1 \restriction X = q$ and $p_5(x) = x$ and $q_1(x) = x$, then $(p_5 \cdot q_1) \restriction X = p \cdot q$ and $(p_5 \cdot q_1)(x) = x$.

(20) For every element t_1 of the permutations of k-element set such that t_1 is a transposition holds $t_1 \cdot t_1 = \mathrm{idseq}(k)$ and $t_1 = {t_1}^{-1}$.

(21) Let given p_1. Then there exists a finite sequence P of elements of A_n such that

(i) $p_1 = \prod P$, and

(ii) for every i such that $i \in \mathrm{dom}\, P$ there exists an element t_2 of the permutations of n-element set such that $P(i) = t_2$ and t_2 is a transposition.

(22) K is Fanoian iff $\mathbf{1}_K \neq -\mathbf{1}_K$.

(23) For every Fanoian field K holds p_2 is even iff $\mathrm{sgn}(p_2, K) = \mathbf{1}_K$ and p_2 is odd iff $\mathrm{sgn}(p_2, K) = -\mathbf{1}_K$.

(24) For all p_3, q_2, p_4 such that $p_4 = p_3 \cdot q_2$ holds $\mathrm{sgn}(p_4, K) = \mathrm{sgn}(p_3, K) \cdot \mathrm{sgn}(q_2, K)$.

(25) p is even and q is even or p is odd and q is odd iff $p \cdot q$ is even.

(26) $(-1)^{\mathrm{sgn}(p_2)} a = \mathrm{sgn}(p_2, K) \cdot a$.

(27) For every element t_1 of the permutations of $(n+2)$-element set such that t_1 is a transposition holds t_1 is odd.

Let us consider n. Observe that there exists a permutation of $\mathrm{Seg}(n+2)$ which is odd.

2. The Determinant of a Linear Combination of Lines

For simplicity, we follow the rules: p_6 denotes a finite sequence of elements of D, M denotes a matrix over D of dimension $n \times m$, p_7, q_3 denote finite sequences of elements of K, and A, B denote matrices over K of dimension n.

Let us consider l, n, m, D, let M be a matrix over D of dimension $n \times m$, and let p_6 be a finite sequence of elements of D. The functor $\mathrm{ReplaceLine}(M, l, p_6)$ yields a matrix over D of dimension $n \times m$ and is defined as follows:

(Def. 3)(i) $\mathrm{len}\,\mathrm{ReplaceLine}(M, l, p_6) = \mathrm{len}\, M$ and $\mathrm{width}\,\mathrm{ReplaceLine}(M, l, p_6) = \mathrm{width}\, M$ and for all i, j such that $\langle i, j \rangle \in$ the indices of M holds if $i \neq l$, then $(\mathrm{ReplaceLine}(M, l, p_6))_{i,j} = M_{i,j}$ and if $i = l$, then $(\mathrm{ReplaceLine}(M, l, p_6))_{l,j} = p_6(j)$ if $\mathrm{len}\, p_6 = \mathrm{width}\, M$,

(ii) $\mathrm{ReplaceLine}(M, l, p_6) = M$, otherwise.

Let us consider l, n, m, D, let M be a matrix over D of dimension $n \times m$, and let p_6 be a finite sequence of elements of D. We introduce $\mathrm{RLine}(M, l, p_6)$ as a synonym of $\mathrm{ReplaceLine}(M, l, p_6)$.

The following propositions are true:

(28) For all l, M, p_6, i such that $i \in \mathrm{Seg}\, n$ holds if $i = l$ and $\mathrm{len}\, p_6 = \mathrm{width}\, M$, then $\mathrm{Line}(\mathrm{RLine}(M, l, p_6), i) = p_6$ and if $i \neq l$, then $\mathrm{Line}(\mathrm{RLine}(M, l, p_6), i) = \mathrm{Line}(M, i)$.

(29) For all M, p_6 such that $\mathrm{len}\, p_6 = \mathrm{width}\, M$ and for every element p' of D^* such that $p_6 = p'$ holds $\mathrm{RLine}(M, l, p_6) = \mathrm{Replace}(M, l, p')$.

(30) $M = \mathrm{RLine}(M, l, \mathrm{Line}(M, l))$.

(31) Let given l, p_7, q_3, p_1. Suppose $l \in \mathrm{Seg}\, n$ and $\mathrm{len}\, p_7 = n$ and $\mathrm{len}\, q_3 = n$. Let M be a matrix over K of dimension n. Then (the multiplication of K) $\circledast$ (p_1-Path $\mathrm{RLine}(M, l, a \cdot p_7 + b \cdot q_3)$) $= a \cdot$ ((the multiplication of K) $\circledast$ (p_1-Path $\mathrm{RLine}(M, l, p_7)$)) $+ b \cdot$ ((the multiplication of K) $\circledast$ (p_1-Path $\mathrm{RLine}(M, l, q_3)$)).

(32) Let given l, p_7, q_3, p_1. Suppose $l \in \mathrm{Seg}\, n$ and $\mathrm{len}\, p_7 = n$ and $\mathrm{len}\, q_3 = n$. Let M be a matrix over K of dimension n. Then (the product on paths of $\mathrm{RLine}(M, l, a \cdot p_7 + b \cdot q_3)$)$(p_1) = a \cdot$ (the product on paths of $\mathrm{RLine}(M, l, p_7)$)$(p_1) + b \cdot$ (the product on paths of $\mathrm{RLine}(M, l, q_3)$)$(p_1)$.

(33) Let given l, p_7, q_3. Suppose $l \in \mathrm{Seg}\, n$ and $\mathrm{len}\, p_7 = n$ and $\mathrm{len}\, q_3 = n$. Let M be a matrix over K of dimension n. Then $\mathrm{Det}\,\mathrm{RLine}(M, l, a \cdot p_7 + b \cdot q_3) = a \cdot \mathrm{Det}\,\mathrm{RLine}(M, l, p_7) + b \cdot \mathrm{Det}\,\mathrm{RLine}(M, l, q_3)$.

(34) If $l \in \mathrm{Seg}\, n$ and $\mathrm{len}\, p_7 = n$, then $\mathrm{Det}\,\mathrm{RLine}(A, l, a \cdot p_7) = a \cdot \mathrm{Det}\,\mathrm{RLine}(A, l, p_7)$.

(35) If $l \in \mathrm{Seg}\, n$, then $\mathrm{Det}\,\mathrm{RLine}(A, l, a \cdot \mathrm{Line}(A, l)) = a \cdot \mathrm{Det}\, A$.

(36) If $l \in \mathrm{Seg}\, n$ and $\mathrm{len}\, p_7 = n$ and $\mathrm{len}\, q_3 = n$, then $\mathrm{Det}\,\mathrm{RLine}(A, l, p_7 + q_3) = \mathrm{Det}\,\mathrm{RLine}(A, l, p_7) + \mathrm{Det}\,\mathrm{RLine}(A, l, q_3)$.

3. The Determinant of a Matrix with Permutated Lines and with a Repeated Line

Let us consider n, m, D, let F be a function from $\mathrm{Seg}\, n$ into $\mathrm{Seg}\, n$, and let M be a matrix over D of dimension $n \times m$. Then $M \cdot F$ is a matrix over D of dimension $n \times m$ and it can be characterized by the condition:

(Def. 4) $\mathrm{len}(M \cdot F) = \mathrm{len}\, M$ and $\mathrm{width}(M \cdot F) = \mathrm{width}\, M$ and for all i, j, k such that $\langle i, j \rangle \in$ the indices of M and $F(i) = k$ holds $(M \cdot F)_{i,j} = M_{k,j}$.

The following propositions are true:

(37)(i) The indices of $M =$ the indices of $M \cdot F$, and

(ii) for all i, j such that $\langle i, j\rangle \in$ the indices of M there exists k such that $F(i) = k$ and $\langle k, j\rangle \in$ the indices of M and $(M \cdot F)_{i,j} = M_{k,j}$.

(38) For every matrix M over D of dimension $n \times m$ and for every F and for every k such that $k \in \operatorname{Seg} n$ holds $\operatorname{Line}(M \cdot F, k) = M(F(k))$.

(39) $M \cdot \operatorname{idseq}(n) = M$.

(40) For all p, P_1, q such that $q = p \cdot {P_1}^{-1}$ holds p-Path $A \cdot P_1 = (q$-Path $A) \cdot P_1$.

(41) For all p, P_1, q such that $q = p \cdot {P_1}^{-1}$ holds (the multiplication of K) $\circledast$ (p-Path $A \cdot P_1$) = (the multiplication of K) $\circledast$ (q-Path A).

(42) For all p_3, q_2 such that $q_2 = {p_3}^{-1}$ holds $\operatorname{sgn}(p_3, K) = \operatorname{sgn}(q_2, K)$.

(43) Let M be a matrix over K of dimension $n+2$ and given p_2, P_2. Suppose $p_2 = P_2$. Let given p_3, q_2. Suppose $q_2 = p_3 \cdot {P_2}^{-1}$. Then (the product on paths of M)$(q_2) = \operatorname{sgn}(p_2, K) \cdot$ (the product on paths of $M \cdot P_2$)(p_3).

(44) Let given p_1. Then there exists a permutation P of the permutations of n-element set such that for every element p of the permutations of n-element set holds $P(p) = p \cdot p_1$.

(45) For every matrix M over K of dimension $n+2 \times n+2$ and for all p_2, P_2 such that $p_2 = P_2$ holds $\operatorname{Det}(M \cdot P_2) = \operatorname{sgn}(p_2, K) \cdot \operatorname{Det} M$.

(46) For every matrix M over K of dimension n and for all p_1, P_1 such that $p_1 = P_1$ holds $\operatorname{Det}(M \cdot P_1) = (-1)^{\operatorname{sgn}(p_1)} \operatorname{Det} M$.

(47) Let P_3 be a permutation of the permutations of n-element set and given p_1. If p_1 is odd and for every p holds $P_3(p) = p \cdot p_1$, then ${P_3}^{\circ}\{p : p$ is even$\} = \{q : q$ is odd$\}$.

(48) Let given n. Suppose $n \geq 2$. Then there exist finite sets O_1, E_1 such that $E_1 = \{p : p$ is even$\}$ and $O_1 = \{q : q$ is odd$\}$ and $E_1 \cap O_1 = \emptyset$ and $E_1 \cup O_1 =$ the permutations of n-element set and $\operatorname{card} E_1 = \operatorname{card} O_1$.

(49) Let given i, j. Suppose $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i < j$. Let M be a matrix over K of dimension n. Suppose $\operatorname{Line}(M, i) = \operatorname{Line}(M, j)$. Let p, q, t_1 be elements of the permutations of n-element set. Suppose $q = p \cdot t_1$ and t_1 is a transposition and $t_1(i) = j$. Then (the product on paths of M)$(q) = -$(the product on paths of M)(p).

(50) Let given i, j. Suppose $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i < j$. Let M be a matrix over K of dimension n. If $\operatorname{Line}(M, i) = \operatorname{Line}(M, j)$, then $\operatorname{Det} M = 0_K$.

(51) For all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i \neq j$ holds $\operatorname{Det} \operatorname{RLine}(A, i, \operatorname{Line}(A, j)) = 0_K$.

(52) For all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i \neq j$ holds $\operatorname{Det} \operatorname{RLine}(A, i, a \cdot \operatorname{Line}(A, j)) = 0_K$.

(53) For all i, j such that $i \in \operatorname{Seg} n$ and $j \in \operatorname{Seg} n$ and $i \neq j$ holds $\operatorname{Det} A =$

Det RLine$(A, i, \mathrm{Line}(A, i) + a \cdot \mathrm{Line}(A, j))$.

(54) If $F \notin$ the permutations of n-element set, then $\mathrm{Det}(A \cdot F) = 0_K$.

4. The Determinant of a Product of Two Square Matrices

Let K be a non empty loop structure. The functor addFinS K yielding a binary operation on (the carrier of K)* is defined as follows:

(Def. 5) For all elements p_5, p_3 of (the carrier of K)* holds $(\mathrm{addFinS}\,K)(p_5, p_3) = p_5 + p_3$.

Let K be an Abelian non empty loop structure. One can verify that addFinS K is commutative.

Let K be an add-associative non empty loop structure. Note that addFinS K is associative.

The following propositions are true:

(55) Let A, B be matrices over K. Suppose width $A = \mathrm{len}\, B$ and $\mathrm{len}\, B > 0$. Let given i. Suppose $i \in \mathrm{Seg}\,\mathrm{len}\, A$. Then there exists a finite sequence P of elements of (the carrier of K)* such that $\mathrm{len}\, P = \mathrm{len}\, B$ and $\mathrm{Line}(A \cdot B, i) = \mathrm{addFinS}\, K \odot P$ and for every j such that $j \in \mathrm{Seg}\,\mathrm{len}\, B$ holds $P(j) = A_{i,j} \cdot \mathrm{Line}(B, j)$.

(56) Let A, B, C be matrices over K of dimension n and given i. Suppose $i \in \mathrm{Seg}\, n$. Then there exists a finite sequence P of elements of K such that $\mathrm{len}\, P = n$ and $\mathrm{Det}\,\mathrm{RLine}(C, i, \mathrm{Line}(A \cdot B, i)) =$ the addition of $K \odot P$ and for every j such that $j \in \mathrm{Seg}\, n$ holds $P(j) = A_{i,j} \cdot \mathrm{Det}\,\mathrm{RLine}(C, i, \mathrm{Line}(B, j))$.

(57) Let X be a set, Y be a non empty set, and given x. Suppose $x \notin X$. Then there exists a function B_1 from $[\!: Y^X, Y :\!]$ into $Y^{X \cup \{x\}}$ such that

(i) B_1 is bijective, and

(ii) for every function f from X into Y and for every function F from $X \cup \{x\}$ into Y such that $F \restriction X = f$ holds $B_1(\langle f, F(x) \rangle) = F$.

(58) Let X be a finite set, Y be a non empty finite set, and given x. Suppose $x \notin X$. Let F be a binary operation on D. Suppose F is commutative and associative and has a unity and an inverse operation. Let f be a function from Y^X into D and g be a function from $Y^{X \cup \{x\}}$ into D. Suppose that for every function H from X into Y and for every element S_1 of $\mathrm{Fin}(Y^{X \cup \{x\}})$ such that $S_1 = \{h; h$ ranges over functions from $X \cup \{x\}$ into $Y\colon h \restriction X = H\}$ holds $F\text{-}\sum_{S_1} g = f(H)$. Then $F\text{-}\sum_{\Omega^{\mathrm{f}}_{Y^X}} f = F\text{-}\sum_{\Omega^{\mathrm{f}}_{Y^{X \cup \{x\}}}} g$.

(59) Let A, B be matrices over D of dimension $n \times m$ and given i. Suppose $i \leq n$ and $0 < n$. Let F be a function from $\mathrm{Seg}\, i$ into $\mathrm{Seg}\, n$. Then there exists a matrix M over D of dimension $n \times m$ such that $M = A {+\cdot} (B \cdot$

(idseq(n)+$\cdot F$))$\restriction$Seg i and for every j holds if $j \in \operatorname{Seg} i$, then $M(j) = B(F(j))$ and if $j \notin \operatorname{Seg} i$, then $M(j) = A(j)$.

(60) Let A, B be matrices over K of dimension n. Suppose $0 < n$. Then there exists a function P from $(\operatorname{Seg} n)^{\operatorname{Seg} n}$ into the carrier of K such that

(i) for every function F from Seg n into Seg n there exists a finite sequence P_4 of elements of K such that len $P_4 = n$ and for all natural numbers F_1, j such that $j \in \operatorname{Seg} n$ and $F_1 = F(j)$ holds $P_4(j) = A_{j,F_1}$ and $P(F) = (($the multiplication of $K) \circledast (P_4)) \cdot \operatorname{Det}(B \cdot F)$, and

(ii) $\operatorname{Det}(A \cdot B) = ($the addition of $K)\text{-}\sum_{\Omega^{\rm f}_{(\operatorname{Seg} n)^{\operatorname{Seg} n}}} P$.

(61) Let A, B be matrices over K of dimension n. Suppose $0 < n$. Then there exists a function P from the permutations of n-element set into the carrier of K such that

(i) $\operatorname{Det}(A \cdot B) = ($the addition of $K)\text{-}\sum_{\Omega^{\rm f}_{\text{the permutations of } n\text{-element set}}} P$, and

(ii) for every element p_1 of the permutations of n-element set holds $P(p_1) = (($the multiplication of $K) \circledast (p_1\text{-Path}\, A)) \cdot (-1)^{\operatorname{sgn}(p_1)} \operatorname{Det} B$.

(62) For all matrices A, B over K of dimension n such that $0 < n$ holds $\operatorname{Det}(A \cdot B) = \operatorname{Det} A \cdot \operatorname{Det} B$.

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