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Tietze Extension Theorem

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Summary. In this paper we formalize the Tietze extension theorem using as a basis the proof presented at the PlanetMath web server[1].

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The articles [24], [26], [1], [2], [23], [11], [4], [21], [27], [5], [28], [7], [6], [17], [16], [22], [18], [20], [19], [25], [9], [10], [13], [14], [8], [12], [3], and [15] provide the notation and terminology for this paper.

We adopt the following rules: r, s denote real numbers, X denotes a set, and f, g, h denote real-yielding functions.

The following propositions are true:

(1) For all real numbers a, b, c such that $|a - b| \leq c$ holds $b - c \leq a$ and $a \leq b + c$.

(2) If $r < s$, then $]-\infty, r]$ misses $[s, +\infty[$.

(3) If $r \leq s$, then $]-\infty, r[$ misses $]s, +\infty[$.

(4) If $f \subseteq g$, then $h - f \subseteq h - g$.

(5) If $f \subseteq g$, then $f - h \subseteq g - h$.

[1] `http://planetmath.org/encyclopedia/ProofOfTietzeExtensionTheorem2.html`

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Let f be a real-yielding function, let r be a real number, and let X be a set. We say that f is absolutely bounded by r in X if and only if:

(Def. 1) For every set x such that $x \in X \cap \operatorname{dom} f$ holds $|f(x)| \leq r$.

Let us mention that there exists a sequence of real numbers which is summable, constant, and convergent.

We now state the proposition

(6) For every empty topological space T_1 and for every topological space T_2 holds every map from T_1 into T_2 is continuous.

Let T_1 be a topological space and let T_2 be a non empty topological space. Observe that there exists a map from T_1 into T_2 which is continuous.

We now state several propositions:

(7) For all summable sequences f, g of real numbers such that for every natural number n holds $f(n) \leq g(n)$ holds $\sum f \leq \sum g$.

(8) For every sequence f of real numbers such that f is absolutely summable holds $|\sum f| \leq \sum |f|$.

(9) Let f be a sequence of real numbers and a, r be positive real numbers. Suppose $r < 1$ and for every natural number n holds $|f(n) - f(n+1)| \leq a \cdot r^n$. Then f is convergent and for every natural number n holds $|\lim f - f(n)| \leq \frac{a \cdot r^n}{1-r}$.

(10) Let f be a sequence of real numbers and a, r be positive real numbers. Suppose $r < 1$ and for every natural number n holds $|f(n) - f(n+1)| \leq a \cdot r^n$. Then $\lim f \geq f(0) - \frac{a}{1-r}$ and $\lim f \leq f(0) + \frac{a}{1-r}$.

(11) Let X, Z be non empty sets and F be a sequence of partial functions from X into $\mathbb{R}$. Suppose Z is common for elements of F. Let a, r be positive real numbers. Suppose $r < 1$ and for every natural number n holds $F(n) - F(n+1)$ is absolutely bounded by $a \cdot r^n$ in Z. Then F is uniform-convergent on Z and for every natural number n holds $\lim_Z F - F(n)$ is absolutely bounded by $\frac{a \cdot r^n}{1-r}$ in Z.

(12) Let X, Z be non empty sets and F be a sequence of partial functions from X into $\mathbb{R}$. Suppose Z is common for elements of F. Let a, r be positive real numbers. Suppose $r < 1$ and for every natural number n holds $F(n) - F(n+1)$ is absolutely bounded by $a \cdot r^n$ in Z. Let z be an element of Z. Then $(\lim_Z F)(z) \geq F(0)(z) - \frac{a}{1-r}$ and $(\lim_Z F)(z) \leq F(0)(z) + \frac{a}{1-r}$.

(13) Let X, Z be non empty sets and F be a sequence of partial functions from X into $\mathbb{R}$. Suppose Z is common for elements of F. Let a, r be positive real numbers and f be a function from Z into $\mathbb{R}$. Suppose $r < 1$ and for every natural number n holds $F(n) - f$ is absolutely bounded by $a \cdot r^n$ in Z. Then F is point-convergent on Z and $\lim_Z F = f$.

Let S, T be topological structures, let A be an empty subset of S, and let f be a map from S into T. Note that $f{\restriction}A$ is empty.

Let T be a topological space and let A be a closed subset of T. Note that $T \restriction A$ is closed.

The following propositions are true:

(14) Let X, Y be non empty topological spaces, X_1, X_2 be non empty subspaces of X, f_1 be a map from X_1 into Y, and f_2 be a map from X_2 into Y. Suppose X_1 misses X_2 or $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$. Let x be a point of X. Then

(i) if $x \in$ the carrier of X_1, then $(f_1 \cup f_2)(x) = f_1(x)$, and

(ii) if $x \in$ the carrier of X_2, then $(f_1 \cup f_2)(x) = f_2(x)$.

(15) Let X, Y be non empty topological spaces, X_1, X_2 be non empty subspaces of X, f_1 be a map from X_1 into Y, and f_2 be a map from X_2 into Y. If X_1 misses X_2 or $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$, then $\mathrm{rng}(f_1 \cup f_2) \subseteq \mathrm{rng}\, f_1 \cup \mathrm{rng}\, f_2$.

(16) Let X, Y be non empty topological spaces, X_1, X_2 be non empty subspaces of X, f_1 be a map from X_1 into Y, and f_2 be a map from X_2 into Y. Suppose X_1 misses X_2 or $f_1 \restriction (X_1 \cap X_2) = f_2 \restriction (X_1 \cap X_2)$. Then for every subset A of X_1 holds $(f_1 \cup f_2)^\circ A = f_1{}^\circ A$ and for every subset A of X_2 holds $(f_1 \cup f_2)^\circ A = f_2{}^\circ A$.

(17) If $f \subseteq g$ and g is absolutely bounded by r in X, then f is absolutely bounded by r in X.

(18) If $X \subseteq \mathrm{dom}\, f$ or $\mathrm{dom}\, g \subseteq \mathrm{dom}\, f$ and if $f \restriction X = g \restriction X$ and if f is absolutely bounded by r in X, then g is absolutely bounded by r in X.

In the sequel T is a non empty topological space and A is a closed subset of T.

One can prove the following propositions:

(19) Suppose $r > 0$ and T is T_4. Let f be a continuous map from $T \restriction A$ into $\mathbb{R}^{\mathbf{1}}$. Suppose f is absolutely bounded by r in A. Then there exists a continuous map g from T into $\mathbb{R}^{\mathbf{1}}$ such that g is absolutely bounded by $\frac{r}{3}$ in $\mathrm{dom}\, g$ and $f - g$ is absolutely bounded by $\frac{2 \cdot r}{3}$ in A.

(20) Suppose that for all non empty closed subsets A, B of T such that A misses B there exists a continuous map f from T into $\mathbb{R}^{\mathbf{1}}$ such that $f^\circ A = \{0\}$ and $f^\circ B = \{1\}$. Then T is a T_4 space.

(21) Let f be a map from T into $\mathbb{R}^{\mathbf{1}}$ and x be a point of T. Then f is continuous at x if and only if for every real number e such that $e > 0$ there exists a subset H of T such that H is open and $x \in H$ and for every point y of T such that $y \in H$ holds $|f(y) - f(x)| < e$.

(22) Let F be a sequence of partial functions from the carrier of T into $\mathbb{R}$. Suppose that

(i) F is uniform-convergent on the carrier of T, and

(ii) for every natural number i holds $F(i)$ is a continuous map from T into

$\mathbb{R}^{\mathbf{1}}$.

Then $\lim_{\text{the carrier of } T} F$ is a continuous map from T into $\mathbb{R}^{\mathbf{1}}$.

(23) Let T be a non empty topological space, f be a map from T into $\mathbb{R}^{\mathbf{1}}$, and r be a positive real number. Then f is absolutely bounded by r in the carrier of T if and only if f is a map from T into $[-r, r]_{\mathrm{T}}$.

(24) If $f-g$ is absolutely bounded by r in X, then $g-f$ is absolutely bounded by r in X.

(25) Suppose T is T_4. Let given A and f be a map from $T \restriction A$ into $[-1, 1]_{\mathrm{T}}$. Suppose f is continuous. Then there exists a continuous map g from T into $[-1, 1]_{\mathrm{T}}$ such that $g \restriction A = f$.

(26) Suppose that for every non empty closed subset A of T and for every continuous map f from $T \restriction A$ into $[-1, 1]_{\mathrm{T}}$ there exists a continuous map g from T into $[-1, 1]_{\mathrm{T}}$ such that $g \restriction A = f$. Then T is T_4.

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