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The Inner Product and Conjugate of Matrix of Complex Numbers

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Summary. Concepts of the inner product and conjugate of matrix of complex numbers are defined here. Operations such as addition, subtraction, scalar multiplication and inner product are introduced using correspondent definitions of the conjugate of a matrix of a complex field. Many equations for such operations consist like a case of the conjugate of matrix of a field and some operations on the set of sum of complex numbers are introduced.

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The papers [20], [24], [18], [25], [7], [8], [9], [3], [19], [2], [4], [11], [5], [10], [6], [17], [1], [13], [14], [23], [12], [15], [16], [22], and [21] provide the notation and terminology for this paper.

We follow the rules: i, j denote natural numbers, a denotes an element of $\mathbb{C}$, and R_1, R_2 denote elements of $\mathbb{C}^i$.

Let M be a matrix over $\mathbb{C}$. The functor $\overline{M}$ yields a matrix over $\mathbb{C}$ and is defined by:

(Def. 1) $\operatorname{len} \overline{M} = \operatorname{len} M$ and $\operatorname{width} \overline{M} = \operatorname{width} M$ and for all natural numbers i, j such that $\langle i, j \rangle \in$ the indices of M holds $\overline{M} \circ (i,j) = \overline{M \circ (i,j)}$.

One can prove the following propositions:

(1) For every matrix M over $\mathbb{C}$ holds $\langle i, j \rangle \in$ the indices of M iff $1 \leq i$ and $i \leq \operatorname{len} M$ and $1 \leq j$ and $j \leq \operatorname{width} M$.

(2) For every matrix M over $\mathbb{C}$ holds $\overline{\overline{M}} = M$.

(3) For every complex number a and for every matrix M over $\mathbb{C}$ holds $\operatorname{len}(a \cdot M) = \operatorname{len} M$ and $\operatorname{width}(a \cdot M) = \operatorname{width} M$.

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(4) Let i, j be natural numbers, a be a complex number, and M be a matrix over $\mathbb{C}$. Suppose $\mathrm{len}(a \cdot M) = \mathrm{len}\, M$ and $\mathrm{width}(a \cdot M) = \mathrm{width}\, M$ and $\langle i, j \rangle \in$ the indices of M. Then $(a \cdot M) \circ (i, j) = a \cdot (M \circ (i, j))$.

(5) For every complex number a and for every matrix M over $\mathbb{C}$ holds $\overline{a \cdot M} = \overline{a} \cdot \overline{M}$.

(6) For all matrices M_1, M_2 over $\mathbb{C}$ holds $\mathrm{len}(M_1 + M_2) = \mathrm{len}\, M_1$ and $\mathrm{width}(M_1 + M_2) = \mathrm{width}\, M_1$.

(7) Let i, j be natural numbers and M_1, M_2 be matrices over $\mathbb{C}$. Suppose $\mathrm{len}\, M_1 = \mathrm{len}\, M_2$ and $\mathrm{width}\, M_1 = \mathrm{width}\, M_2$ and $\langle i, j \rangle \in$ the indices of M_1. Then $(M_1 + M_2) \circ (i, j) = (M_1 \circ (i, j)) + (M_2 \circ (i, j))$.

(8) For all matrices M_1, M_2 over $\mathbb{C}$ such that $\mathrm{len}\, M_1 = \mathrm{len}\, M_2$ and $\mathrm{width}\, M_1 = \mathrm{width}\, M_2$ holds $\overline{M_1 + M_2} = \overline{M_1} + \overline{M_2}$.

(9) For every matrix M over $\mathbb{C}$ holds $\mathrm{len}(-M) = \mathrm{len}\, M$ and $\mathrm{width}(-M) = \mathrm{width}\, M$.

(10) Let i, j be natural numbers and M be a matrix over $\mathbb{C}$. If $\mathrm{len}(-M) = \mathrm{len}\, M$ and $\mathrm{width}(-M) = \mathrm{width}\, M$ and $\langle i, j \rangle \in$ the indices of M, then $(-M) \circ (i, j) = -(M \circ (i, j))$.

(11) For every matrix M over $\mathbb{C}$ holds $(-1) \cdot M = -M$.

(12) For every matrix M over $\mathbb{C}$ holds $\overline{-M} = -\overline{M}$.

(13) For all matrices M_1, M_2 over $\mathbb{C}$ holds $\mathrm{len}(M_1 - M_2) = \mathrm{len}\, M_1$ and $\mathrm{width}(M_1 - M_2) = \mathrm{width}\, M_1$.

(14) Let i, j be natural numbers and M_1, M_2 be matrices over $\mathbb{C}$. Suppose $\mathrm{len}\, M_1 = \mathrm{len}\, M_2$ and $\mathrm{width}\, M_1 = \mathrm{width}\, M_2$ and $\langle i, j \rangle \in$ the indices of M_1. Then $(M_1 - M_2) \circ (i, j) = (M_1 \circ (i, j)) - (M_2 \circ (i, j))$.

(15) For all matrices M_1, M_2 over $\mathbb{C}$ such that $\mathrm{len}\, M_1 = \mathrm{len}\, M_2$ and $\mathrm{width}\, M_1 = \mathrm{width}\, M_2$ holds $\overline{M_1 - M_2} = \overline{M_1} - \overline{M_2}$.

Let M be a matrix over $\mathbb{C}$. The functor M^* yields a matrix over $\mathbb{C}$ and is defined by:

(Def. 2) $M^* = \overline{M^{\mathrm{T}}}$.

Let x be a finite sequence of elements of $\mathbb{C}$. Let us assume that $\mathrm{len}\, x > 0$. The functor FinSeq2Matrix x yielding a matrix over $\mathbb{C}$ is defined as follows:

(Def. 3) $\mathrm{len}\,\mathrm{FinSeq2Matrix}\, x = \mathrm{len}\, x$ and $\mathrm{width}\,\mathrm{FinSeq2Matrix}\, x = 1$ and for every j such that $j \in \mathrm{Seg}\,\mathrm{len}\, x$ holds $(\mathrm{FinSeq2Matrix}\, x)(j) = \langle x(j) \rangle$.

Let M be a matrix over $\mathbb{C}$. The functor Matrix2FinSeq M yields a finite sequence of elements of $\mathbb{C}$ and is defined as follows:

(Def. 4) $\mathrm{Matrix2FinSeq}\, M = M_{\square,1}$.

Let F_1, F_2 be finite sequences of elements of $\mathbb{C}$. The functor $F_1 \bullet F_2$ yielding a finite sequence of elements of $\mathbb{C}$ is defined as follows:

(Def. 5) $F_1 \bullet F_2 = (\cdot_{\mathbb{C}})^{\circ}(F_1, F_2)$.

Let us observe that the functor $F_1 \bullet F_2$ is commutative.

Let F be a finite sequence of elements of $\mathbb{C}$. The functor $\sum F$ yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 6) $\sum F = +_{\mathbb{C}} \circledast F$.

Let M be a matrix over $\mathbb{C}$ and let F be a finite sequence of elements of $\mathbb{C}$. The functor $M \cdot F$ yielding a finite sequence of elements of $\mathbb{C}$ is defined as follows:

(Def. 7) $\mathrm{len}(M \cdot F) = \mathrm{len}\, M$ and for every i such that $i \in \mathrm{Seg}\,\mathrm{len}\, M$ holds $(M \cdot F)(i) = \sum(\mathrm{Line}(M, i) \bullet F)$.

We now state the proposition

(16) $a \cdot (R_1 \bullet R_2) = a \cdot R_1 \bullet R_2$.

Let M be a matrix over $\mathbb{C}$ and let a be a complex number. The functor $M \cdot a$ yielding a matrix over $\mathbb{C}$ is defined by:

(Def. 8) $M \cdot a = a \cdot M$.

We now state three propositions:

(17) For every element a of $\mathbb{C}$ and for every matrix M over $\mathbb{C}$ holds $\overline{M \cdot a} = \overline{a} \cdot \overline{M}$.

(18) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\mathrm{len}\, x = \mathrm{len}\, y$ holds $\mathrm{len}(x \bullet y) = \mathrm{len}\, x$ and $\mathrm{len}(x \bullet y) = \mathrm{len}\, y$.

(19) Let F_1, F_2 be finite sequences of elements of $\mathbb{C}$ and i be a natural number. If $i \in \mathrm{dom}(F_1 \bullet F_2)$, then $(F_1 \bullet F_2)(i) = F_1(i) \cdot F_2(i)$.

Let us consider i, R_1, R_2. Then $R_1 \bullet R_2$ is an element of $\mathbb{C}^i$.

We now state a number of propositions:

(20) $(R_1 \bullet R_2)(j) = R_1(j) \cdot R_2(j)$.

(21) For all elements a, b of $\mathbb{C}$ holds $\overline{+_{\mathbb{C}}(a, \overline{b})} = +_{\mathbb{C}}(\overline{a}, b)$.

(22) Let F be a finite sequence of elements of $\mathbb{C}$. Then there exists a function G from $\mathbb{N}$ into $\mathbb{C}$ such that for every natural number n if $1 \leq n$ and $n \leq \mathrm{len}\, F$, then $G(n) = F(n)$.

(23) For every finite sequence F of elements of $\mathbb{C}$ such that $\mathrm{len}\, \overline{F} \geq 1$ holds $+_{\mathbb{C}} \circledast \overline{F} = \overline{+_{\mathbb{C}} \circledast F}$.

(24) For every finite sequence F of elements of $\mathbb{C}$ such that $\mathrm{len}\, F \geq 1$ holds $\sum \overline{F} = \overline{\sum F}$.

(25) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\mathrm{len}\, x = \mathrm{len}\, y$ holds $\overline{x \bullet \overline{y}} = y \bullet \overline{x}$.

(26) For all finite sequences x, y of elements of $\mathbb{C}$ and for every element a of $\mathbb{C}$ such that $\mathrm{len}\, x = \mathrm{len}\, y$ holds $x \bullet a \cdot y = a \cdot (x \bullet y)$.

(27) For all finite sequences x, y of elements of $\mathbb{C}$ and for every element a of $\mathbb{C}$ such that $\mathrm{len}\, x = \mathrm{len}\, y$ holds $a \cdot x \bullet y = a \cdot (x \bullet y)$.

(28) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $\overline{x \bullet y} = \overline{x} \bullet \overline{y}$.

(29) For every finite sequence F of elements of $\mathbb{C}$ and for every element a of $\mathbb{C}$ holds $\sum(a \cdot F) = a \cdot \sum F$.

Let x be a finite sequence of elements of $\mathbb{R}$. The functor $\mathrm{FR2FC}\, x$ yielding a finite sequence of elements of $\mathbb{C}$ is defined as follows:

(Def. 9) $\mathrm{FR2FC}\, x = x$.

Next we state a number of propositions:

(30) Let R be a finite sequence of elements of $\mathbb{R}$ and F be a finite sequence of elements of $\mathbb{C}$. If $R = F$ and $\operatorname{len} R \geq 1$, then $+_{\mathbb{R}} \circledast R = +_{\mathbb{C}} \circledast F$.

(31) Let x be a finite sequence of elements of $\mathbb{R}$ and y be a finite sequence of elements of $\mathbb{C}$. If $x = y$ and $\operatorname{len} x \geq 1$, then $\sum x = \sum y$.

(32) For all finite sequences F_1, F_2 of elements of $\mathbb{C}$ such that $\operatorname{len} F_1 = \operatorname{len} F_2$ holds $\sum(F_1 - F_2) = \sum F_1 - \sum F_2$.

(33) Let F_1, F_2 be finite sequences of elements of $\mathbb{C}$ and i be a natural number. If $i \in \operatorname{dom}(F_1 + F_2)$, then $(F_1 + F_2)(i) = F_1(i) + F_2(i)$.

(34) Let F_1, F_2 be finite sequences of elements of $\mathbb{C}$ and i be a natural number. If $i \in \operatorname{dom}(F_1 - F_2)$, then $(F_1 - F_2)(i) = F_1(i) - F_2(i)$.

(35) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $(x - y) \bullet z = x \bullet z - y \bullet z$.

(36) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $x \bullet (y - z) = x \bullet y - x \bullet z$.

(37) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $x \bullet (y + z) = x \bullet y + x \bullet z$.

(38) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $(x + y) \bullet z = x \bullet z + y \bullet z$.

(39) For all finite sequences F_1, F_2 of elements of $\mathbb{C}$ such that $\operatorname{len} F_1 = \operatorname{len} F_2$ holds $\sum(F_1 + F_2) = \sum F_1 + \sum F_2$.

(40) Let x_1, y_1 be finite sequences of elements of $\mathbb{C}$ and x_2, y_2 be finite sequences of elements of $\mathbb{R}$. If $x_1 = x_2$ and $y_1 = y_2$ and $\operatorname{len} x_1 = \operatorname{len} y_2$, then $(\cdot_{\mathbb{C}})^{\circ}(x_1, y_1) = (\cdot_{\mathbb{R}})^{\circ}(x_2, y_2)$.

(41) For all finite sequences x, y of elements of $\mathbb{R}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $\mathrm{FR2FC}(x \bullet y) = \mathrm{FR2FC}\, x \bullet \mathrm{FR2FC}\, y$.

(42) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} x > 0$ holds $|(x, y)| = \sum(x \bullet \overline{y})$.

(43) For all matrices A, B over $\mathbb{C}$ such that $\operatorname{len} A = \operatorname{len} B$ and $\operatorname{width} A = \operatorname{width} B$ holds the indices of $A =$ the indices of B.

(44) Let i, j be natural numbers and M_1, M_2 be matrices over $\mathbb{C}$. If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and $j \in \operatorname{Seg} \operatorname{len} M_1$, then $\operatorname{Line}(M_1 +$

$M_2, j) = \mathrm{Line}(M_1, j) + \mathrm{Line}(M_2, j)$.

(45) For every matrix M over $\mathbb{C}$ such that $i \in \mathrm{Seg}\,\mathrm{len}\,M$ holds $\overline{\mathrm{Line}(M, i)} = \mathrm{Line}(\overline{M}, i)$.

(46) Let F be a finite sequence of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If $\mathrm{len}\,F = \mathrm{width}\,M$, then $F \bullet \overline{\mathrm{Line}(\overline{M}, i)} = \overline{\mathrm{Line}(\overline{M}, i) \bullet \overline{F}}$.

(47) Let F be a finite sequence of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If $\mathrm{len}\,F = \mathrm{width}\,M$ and $\mathrm{len}\,F \geq 1$, then $\overline{M \cdot F} = \overline{M} \cdot \overline{F}$.

(48) For all finite sequences F_1, F_2, F_3 of elements of $\mathbb{C}$ such that $\mathrm{len}\,F_1 = \mathrm{len}\,F_2$ and $\mathrm{len}\,F_2 = \mathrm{len}\,F_3$ holds $F_1 \bullet (F_2 \bullet F_3) = (F_1 \bullet F_2) \bullet F_3$.

(49) For every finite sequence F of elements of $\mathbb{C}$ holds $\sum(-F) = -\sum F$.

(50) For every element z of $\mathbb{C}$ holds $\sum\langle z\rangle = z$.

(51) For all finite sequences F_1, F_2 of elements of $\mathbb{C}$ holds $\sum(F_1 \frown F_2) = \sum F_1 + \sum F_2$.

Let M be a matrix over $\mathbb{C}$. The functor $\mathrm{LineSum}\,M$ yielding a finite sequence of elements of $\mathbb{C}$ is defined as follows:

(Def. 10) $\mathrm{len}\,\mathrm{LineSum}\,M = \mathrm{len}\,M$ and for every natural number i such that $i \in \mathrm{Seg}\,\mathrm{len}\,M$ holds $(\mathrm{LineSum}\,M)(i) = \sum \mathrm{Line}(M, i)$.

Let M be a matrix over $\mathbb{C}$. The functor $\mathrm{ColSum}\,M$ yielding a finite sequence of elements of $\mathbb{C}$ is defined by:

(Def. 11) $\mathrm{len}\,\mathrm{ColSum}\,M = \mathrm{width}\,M$ and for every natural number j such that $j \in \mathrm{Seg}\,\mathrm{width}\,M$ holds $(\mathrm{ColSum}\,M)(j) = \sum(M_{\square,j})$.

Next we state three propositions:

(52) For every finite sequence F of elements of $\mathbb{C}$ such that $\mathrm{len}\,F = 1$ holds $\sum F = F(1)$.

(53) Let f, g be finite sequences of elements of $\mathbb{C}$ and n be a natural number. If $\mathrm{len}\,f = n + 1$ and $g = f \restriction n$, then $\sum f = \sum g + f_{\mathrm{len}\,f}$.

(54) For every matrix M over $\mathbb{C}$ such that $\mathrm{len}\,M > 0$ holds $\sum \mathrm{LineSum}\,M = \sum \mathrm{ColSum}\,M$.

Let M be a matrix over $\mathbb{C}$. The functor $\mathrm{SumAll}\,M$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 12) $\mathrm{SumAll}\,M = \sum \mathrm{LineSum}\,M$.

Next we state two propositions:

(55) For every matrix M over $\mathbb{C}$ holds $\mathrm{ColSum}\,M = \mathrm{LineSum}(M^{\mathrm{T}})$.

(56) For every matrix M over $\mathbb{C}$ such that $\mathrm{len}\,M > 0$ holds $\mathrm{SumAll}\,M = \mathrm{SumAll}(M^{\mathrm{T}})$.

Let x, y be finite sequences of elements of $\mathbb{C}$ and let M be a matrix over $\mathbb{C}$. Let us assume that $\mathrm{len}\,x = \mathrm{len}\,M$ and $\mathrm{len}\,y = \mathrm{width}\,M$. The functor $\mathrm{QuadraticForm}(x, M, y)$ yielding a matrix over $\mathbb{C}$ is defined by the conditions (Def. 13).

(Def. 13)(i) len QuadraticForm$(x, M, y) = $ len x,

(ii) width QuadraticForm$(x, M, y) = $ len y, and

(iii) for all natural numbers i, j such that $\langle i, j \rangle \in$ the indices of M holds QuadraticForm$(x, M, y) \circ (i, j) = x(i) \cdot (M \circ (i, j)) \cdot \overline{y(j)}$.

The following propositions are true:

(57) Let x, y be finite sequences of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If len $x = $ len M and len $y = $ width M and len $x > 0$ and len $y > 0$, then $(\mathrm{QuadraticForm}(x, M, y))^{\mathrm{T}} = \overline{\mathrm{QuadraticForm}(y, M^*, x)}$.

(58) Let x, y be finite sequences of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If len $x = $ len M and len $y = $ width M, then $\overline{\mathrm{QuadraticForm}(x, M, y)} = \mathrm{QuadraticForm}(\overline{x}, \overline{M}, \overline{y})$.

(59) For all finite sequences x, y of elements of $\mathbb{C}$ such that len $x = $ len y and $0 <$ len y holds $|(x, y)| = \overline{|(y, x)|}$.

(60) For all finite sequences x, y of elements of $\mathbb{C}$ such that len $x = $ len y and $0 <$ len y holds $\overline{|(x, y)|} = |(\overline{x}, \overline{y})|$.

(61) For every matrix M over $\mathbb{C}$ such that width $M > 0$ holds $\overline{M^{\mathrm{T}}} = \overline{M}^{\mathrm{T}}$.

(62) Let x, y be finite sequences of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If len $x = $ width M and len $y = $ len M and len $x > 0$ and len $y > 0$, then $|(x, M^* \cdot y)| = $ SumAll QuadraticForm(x, M^{T}, y).

(63) Let x, y be finite sequences of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If len $y = $ len M and len $x = $ width M and len $x > 0$ and len $y > 0$ and len $M > 0$, then $|(M \cdot x, y)| = $ SumAll QuadraticForm(x, M^{T}, y).

(64) Let x, y be finite sequences of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If len $x = $ width M and len $y = $ len M and width $M > 0$ and len $M > 0$, then $|(M \cdot x, y)| = |(x, M^* \cdot y)|$.

(65) Let x, y be finite sequences of elements of $\mathbb{C}$ and M be a matrix over $\mathbb{C}$. If len $x = $ len M and len $y = $ width M and width $M > 0$ and len $M > 0$ and len $x > 0$, then $|(x, M \cdot y)| = |(M^* \cdot x, y)|$.

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