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Generalized Full Adder Circuits (GFAs). Part I

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Summary. In the article we formalized the concept of the Generalized Full Addition and Subtraction circuits (GFAs), defined the structures of calculation units for the redundant signed digit (RSD) operations, and proved the stability of the circuits. Generally, 1-bit binary full adder assumes positive weights to all of its three binary inputs and two outputs. We obtained four type of 1-bit GFA to constract the RSD arithmetic logical units that we generalized full adder to have both positive and negative weights to inputs and outputs.

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The articles [15], [14], [18], [13], [1], [21], [5], [6], [7], [2], [4], [16], [20], [8], [12], [17], [11], [10], [9], [3], and [19] provide the terminology and notation for this paper.

1. Preliminaries

In this article we present several logical schemes. The scheme *1AryBooleEx* deals with a unary functor $\mathcal{F}$ yielding an element of *Boolean*, and states that:

There exists a function f from $Boolean^1$ into *Boolean* such that for every element x of *Boolean* holds $f(\langle x \rangle) = \mathcal{F}(x)$

for all values of the parameter.

The scheme *1AryBooleUniq* deals with a unary functor $\mathcal{F}$ yielding an element of *Boolean*, and states that:

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Let f_1, f_2 be functions from $Boolean^1$ into *Boolean*. Suppose for every element x of *Boolean* holds $f_1(\langle x \rangle) = \mathcal{F}(x)$ and for every element x of *Boolean* holds $f_2(\langle x \rangle) = \mathcal{F}(x)$. Then $f_1 = f_2$

for all values of the parameter.

The scheme *1AryBooleDef* deals with a unary functor $\mathcal{F}$ yielding an element of *Boolean*, and states that:

(i) There exists a function f from $Boolean^1$ into *Boolean* such that for every element x of *Boolean* holds $f(\langle x \rangle) = \mathcal{F}(x)$, and

(ii) for all functions f_1, f_2 from $Boolean^1$ into *Boolean* such that for every element x of *Boolean* holds $f_1(\langle x \rangle) = \mathcal{F}(x)$ and for every element x of *Boolean* holds $f_2(\langle x \rangle) = \mathcal{F}(x)$ holds $f_1 = f_2$

for all values of the parameter.

The function inv1 from $Boolean^1$ into *Boolean* is defined by:

(Def. 1) For every element x of *Boolean* holds $(\mathrm{inv1})(\langle x \rangle) = \neg x$.

Next we state the proposition

(1) For every element x of *Boolean* holds $(\mathrm{inv1})(\langle x \rangle) = \neg x$ and $(\mathrm{inv1})(\langle x \rangle) = \mathrm{nand}_2(\langle x, x \rangle)$ and $(\mathrm{inv1})(\langle 0 \rangle) = 1$ and $(\mathrm{inv1})(\langle 1 \rangle) = 0$.

The function buf1 from $Boolean^1$ into *Boolean* is defined by:

(Def. 2) For every element x of *Boolean* holds $(\mathrm{buf1})(\langle x \rangle) = x$.

One can prove the following proposition

(2) For every element x of *Boolean* holds $(\mathrm{buf1})(\langle x \rangle) = x$ and $(\mathrm{buf1})(\langle x \rangle) = \mathrm{and}_2(\langle x, x \rangle)$ and $(\mathrm{buf1})(\langle 0 \rangle) = 0$ and $(\mathrm{buf1})(\langle 1 \rangle) = 1$.

The function and2c from $Boolean^2$ into *Boolean* is defined by:

(Def. 3) For all elements x, y of *Boolean* holds $(\mathrm{and2c})(\langle x, y \rangle) = x \wedge \neg y$.

Next we state the proposition

(3) Let x, y be elements of *Boolean*. Then $(\mathrm{and2c})(\langle x, y \rangle) = x \wedge \neg y$ and $(\mathrm{and2c})(\langle x, y \rangle) = (\mathrm{and}_{2a})(\langle y, x \rangle)$ and $(\mathrm{and2c})(\langle x, y \rangle) = (\mathrm{nor}_{2a})(\langle x, y \rangle)$ and $(\mathrm{and2c})(\langle 0, 0 \rangle) = 0$ and $(\mathrm{and2c})(\langle 0, 1 \rangle) = 0$ and $(\mathrm{and2c})(\langle 1, 0 \rangle) = 1$ and $(\mathrm{and2c})(\langle 1, 1 \rangle) = 0$.

The function xor2c from $Boolean^2$ into *Boolean* is defined by:

(Def. 4) For all elements x, y of *Boolean* holds $(\mathrm{xor2c})(\langle x, y \rangle) = x \oplus \neg y$.

We now state several propositions:

(4) Let x, y be elements of *Boolean*. Then $(\mathrm{xor2c})(\langle x, y \rangle) = x \oplus \neg y$ and $(\mathrm{xor2c})(\langle x, y \rangle) = (\mathrm{xor}_{2a})(\langle x, y \rangle)$ and $(\mathrm{xor2c})(\langle x, y \rangle) = \mathrm{or}_2(\langle (\mathrm{and}_{2b})(\langle x, y \rangle), \mathrm{and}_2(\langle x, y \rangle) \rangle)$ and $(\mathrm{xor2c})(\langle 0, 0 \rangle) = 1$ and $(\mathrm{xor2c})(\langle 0, 1 \rangle) = 0$ and $(\mathrm{xor2c})(\langle 1, 0 \rangle) = 0$ and $(\mathrm{xor2c})(\langle 1, 1 \rangle) = 1$.

(5) For all elements x, y of *Boolean* holds $\neg(x \oplus y) = \neg x \oplus y$ and $\neg(x \oplus y) = x \oplus \neg y$ and $\neg x \oplus \neg y = x \oplus y$.

(6) For all elements x, y of *Boolean* holds $(\mathrm{inv1})(\langle \mathrm{xor}_2(\langle x, y\rangle)\rangle) = (\mathrm{xor}_{2a})(\langle x, y\rangle)$ and $(\mathrm{inv1})(\langle \mathrm{xor}_2(\langle x, y\rangle)\rangle) = (\mathrm{xor2c})(\langle x, y\rangle)$ and $\mathrm{xor}_2(\langle (\mathrm{inv1})(\langle x\rangle), (\mathrm{inv1})(\langle y\rangle)\rangle) = \mathrm{xor}_2(\langle x, y\rangle)$.

(7) For all elements x, y, z of *Boolean* holds $\neg(x \oplus \neg y \oplus z) = x \oplus \neg y \oplus \neg z$.

(8) For all elements x, y, z of *Boolean* holds $(\mathrm{inv1})(\langle \mathrm{xor}_2(\langle (\mathrm{xor2c})(\langle x, y\rangle), z\rangle)\rangle) = (\mathrm{xor2c})(\langle (\mathrm{xor2c})(\langle x, y\rangle), z\rangle)$.

(9) For all elements x, y, z of *Boolean* holds $\neg x \oplus y \oplus \neg z = x \oplus \neg y \oplus \neg z$.

(10) For all elements x, y, z of *Boolean* holds $(\mathrm{xor2c})(\langle (\mathrm{xor}_{2a})(\langle x, y\rangle), z\rangle) = (\mathrm{xor2c})(\langle (\mathrm{xor2c})(\langle x, y\rangle), z\rangle)$.

(11) For all elements x, y, z of *Boolean* holds $\neg(\neg x \oplus \neg y \oplus \neg z) = x \oplus y \oplus z$.

(12) For all elements x, y, z of *Boolean* holds $(\mathrm{inv1})(\langle (\mathrm{xor2c})(\langle (\mathrm{xor}_{2b})(\langle x, y\rangle), z\rangle)\rangle) = \mathrm{xor}_2(\langle \mathrm{xor}_2(\langle x, y\rangle), z\rangle)$.

2. Generalized Full Adder (GFA) Circuit (TYPE-0)

Let x, y, z be sets. The functor $\mathrm{GFA0CarryIStr}(x, y, z)$ yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined by:

(Def. 5) $\mathrm{GFA0CarryIStr}(x, y, z) = \mathrm{1GateCircStr}(\langle x, y\rangle, \mathrm{and}_2) +\cdot \mathrm{1GateCircStr}(\langle y, z\rangle, \mathrm{and}_2) +\cdot \mathrm{1GateCircStr}(\langle z, x\rangle, \mathrm{and}_2)$.

Let x, y, z be sets. The functor $\mathrm{GFA0CarryICirc}(x, y, z)$ yields a strict Boolean circuit of $\mathrm{GFA0CarryIStr}(x, y, z)$ with denotation held in gates and is defined as follows:

(Def. 6) $\mathrm{GFA0CarryICirc}(x, y, z) = \mathrm{1GateCircuit}(x, y, \mathrm{and}_2) +\cdot \mathrm{1GateCircuit}(y, z, \mathrm{and}_2) +\cdot \mathrm{1GateCircuit}(z, x, \mathrm{and}_2)$.

Let x, y, z be sets. The functor $\mathrm{GFA0CarryStr}(x, y, z)$ yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined as follows:

(Def. 7) $\mathrm{GFA0CarryStr}(x, y, z) = \mathrm{GFA0CarryIStr}(x, y, z) +\cdot \mathrm{1GateCircStr}(\langle \boldsymbol{\langle} \langle x, y\rangle, \mathrm{and}_2 \boldsymbol{\rangle}, \boldsymbol{\langle} \langle y, z\rangle, \mathrm{and}_2 \boldsymbol{\rangle}, \boldsymbol{\langle} \langle z, x\rangle, \mathrm{and}_2 \boldsymbol{\rangle}\rangle, \mathrm{or}_3)$.

Let x, y, z be sets. The functor $\mathrm{GFA0CarryCirc}(x, y, z)$ yields a strict Boolean circuit of $\mathrm{GFA0CarryStr}(x, y, z)$ with denotation held in gates and is defined as follows:

(Def. 8) $\mathrm{GFA0CarryCirc}(x, y, z) = \mathrm{GFA0CarryICirc}(x, y, z) +\cdot \mathrm{1GateCircuit}(\boldsymbol{\langle} \langle x, y\rangle, \mathrm{and}_2 \boldsymbol{\rangle}, \boldsymbol{\langle} \langle y, z\rangle, \mathrm{and}_2 \boldsymbol{\rangle}, \boldsymbol{\langle} \langle z, x\rangle, \mathrm{and}_2 \boldsymbol{\rangle}, \mathrm{or}_3)$.

Let x, y, z be sets. The functor $\mathrm{GFA0CarryOutput}(x, y, z)$ yielding an element of $\mathrm{InnerVertices}(\mathrm{GFA0CarryStr}(x, y, z))$ is defined as follows:

(Def. 9) $\mathrm{GFA0CarryOutput}(x, y, z) = \boldsymbol{\langle} \langle \boldsymbol{\langle} \langle x, y\rangle, \mathrm{and}_2 \boldsymbol{\rangle}, \boldsymbol{\langle} \langle y, z\rangle, \mathrm{and}_2 \boldsymbol{\rangle}, \boldsymbol{\langle} \langle z, x\rangle, \mathrm{and}_2 \boldsymbol{\rangle}\rangle, \mathrm{or}_3 \boldsymbol{\rangle}$.

One can prove the following propositions:

(13) For all sets x, y, z holds InnerVertices(GFA0CarryIStr(x,y,z)) $= \{\langle\langle x, y\rangle, \mathrm{and}_2\rangle, \langle\langle y,z\rangle, \mathrm{and}_2\rangle, \langle\langle z,x\rangle, \mathrm{and}_2\rangle\}$.

(14) For all sets x, y, z holds InnerVertices(GFA0CarryStr(x,y,z)) $= \{\langle\langle x, y\rangle, \mathrm{and}_2\rangle, \langle\langle y,z\rangle, \mathrm{and}_2\rangle, \langle\langle z,x\rangle, \mathrm{and}_2\rangle\} \cup \{$GFA0CarryOutput$(x,y,z)\}$.

(15) For all sets x, y, z holds InnerVertices(GFA0CarryStr(x,y,z)) is a binary relation.

(16) For all sets x, y, z such that $x \neq \langle\langle y,z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z,x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x,y\rangle, \mathrm{and}_2\rangle$ holds InputVertices(GFA0CarryIStr(x,y,z)) $= \{x,y,z\}$.

(17) For all sets x, y, z such that $x \neq \langle\langle y,z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z,x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x,y\rangle, \mathrm{and}_2\rangle$ holds InputVertices(GFA0CarryStr(x,y,z)) $= \{x,y,z\}$.

(18) For all non pair sets x, y, z holds InputVertices(GFA0CarryStr(x,y,z)) has no pairs.

(19) Let x, y, z be sets. Then $x \in$ the carrier of GFA0CarryStr(x,y,z) and $y \in$ the carrier of GFA0CarryStr(x,y,z) and $z \in$ the carrier of GFA0CarryStr(x,y,z) and $\langle\langle x,y\rangle, \mathrm{and}_2\rangle \in$ the carrier of GFA0CarryStr(x,y,z) and $\langle\langle y,z\rangle, \mathrm{and}_2\rangle \in$ the carrier of GFA0CarryStr(x,y,z) and $\langle\langle z,x\rangle, \mathrm{and}_2\rangle \in$ the carrier of GFA0CarryStr(x,y,z) and $\langle\langle\langle\langle x,y\rangle, \mathrm{and}_2\rangle, \langle\langle y,z\rangle, \mathrm{and}_2\rangle, \langle\langle z,x\rangle, \mathrm{and}_2\rangle\rangle, \mathrm{or}_3\rangle \in$ the carrier of GFA0CarryStr(x,y,z).

(20) For all sets x, y, z holds $\langle\langle x,y\rangle, \mathrm{and}_2\rangle \in$ InnerVertices(GFA0CarryStr(x, y,z)) and $\langle\langle y,z\rangle, \mathrm{and}_2\rangle \in$ InnerVertices(GFA0CarryStr(x,y,z)) and $\langle\langle z, x\rangle, \mathrm{and}_2\rangle \in$ InnerVertices(GFA0CarryStr(x,y,z)) and GFA0CarryOutput $(x,y,z) \in$ InnerVertices(GFA0CarryStr(x,y,z)).

(21) For all sets x, y, z such that $x \neq \langle\langle y,z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z,x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x,y\rangle, \mathrm{and}_2\rangle$ holds $x \in$ InputVertices(GFA0CarryStr(x,y,z)) and $y \in$ InputVertices(GFA0CarryStr(x,y,z)) and $z \in$ InputVertices(GFA0CarryStr(x,y,z)).

(22) For all non pair sets x, y, z holds InputVertices(GFA0CarryStr(x,y,z)) $= \{x,y,z\}$.

(23) Let x, y, z be sets, s be a state of GFA0CarryCirc(x,y,z), and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following(s))$(\langle\langle x,y\rangle, \mathrm{and}_2\rangle) = a_1 \wedge a_2$ and (Following(s))$(\langle\langle y,z\rangle, \mathrm{and}_2\rangle) = a_2 \wedge a_3$ and (Following(s))$(\langle\langle z,x\rangle, \mathrm{and}_2\rangle) = a_3 \wedge a_1$.

(24) Let x, y, z be sets, s be a state of GFA0CarryCirc(x,y,z), and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(\langle\langle x,y\rangle, \mathrm{and}_2\rangle)$ and $a_2 = s(\langle\langle y,z\rangle, \mathrm{and}_2\rangle)$ and $a_3 = s(\langle\langle z,x\rangle, \mathrm{and}_2\rangle)$, then (Following$(s)$)(GFA0CarryOutput$(x,y,z)$) $= a_1 \vee a_2 \vee a_3$.

(25) Let x, y, z be sets. Suppose $x \neq \langle\langle y, z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_2\rangle$. Let s be a state of GFA0CarryCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA0CarryOutput(x, y, z)) $= a_1 \wedge a_2 \vee a_2 \wedge a_3 \vee a_3 \wedge a_1$ and (Following$(s, 2)$)($\langle\langle x, y\rangle, \mathrm{and}_2\rangle$) $= a_1 \wedge a_2$ and (Following$(s, 2)$)($\langle\langle y, z\rangle, \mathrm{and}_2\rangle$) $= a_2 \wedge a_3$ and (Following$(s, 2)$)($\langle\langle z, x\rangle, \mathrm{and}_2\rangle$) $= a_3 \wedge a_1$.

(26) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_2\rangle$ and for every state s of GFA0CarryCirc(x, y, z) holds Following$(s, 2)$ is stable.

Let x, y, z be sets. The functor GFA0AdderStr(x, y, z) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined as follows:

(Def. 10) GFA0AdderStr(x, y, z) = 2GatesCircStr$(x, y, z, \mathrm{xor}_2)$.

Let x, y, z be sets. The functor GFA0AdderCirc(x, y, z) yielding a strict Boolean circuit of GFA0AdderStr(x, y, z) with denotation held in gates is defined by:

(Def. 11) GFA0AdderCirc(x, y, z) = 2GatesCircuit$(x, y, z, \mathrm{xor}_2)$.

Let x, y, z be sets. The functor GFA0AdderOutput(x, y, z) yielding an element of InnerVertices(GFA0AdderStr(x, y, z)) is defined by:

(Def. 12) GFA0AdderOutput(x, y, z) = 2GatesCircOutput$(x, y, z, \mathrm{xor}_2)$.

Next we state a number of propositions:

(27) For all sets x, y, z holds InnerVertices(GFA0AdderStr(x, y, z)) $= \{\langle\langle x, y\rangle, \mathrm{xor}_2\rangle\} \cup \{$GFA0AdderOutput$(x, y, z)\}$.

(28) For all sets x, y, z holds InnerVertices(GFA0AdderStr(x, y, z)) is a binary relation.

(29) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ holds InputVertices(GFA0AdderStr(x, y, z)) $= \{x, y, z\}$.

(30) For all non pair sets x, y, z holds InputVertices(GFA0AdderStr(x, y, z)) has no pairs.

(31) Let x, y, z be sets. Then
 (i) $x \in$ the carrier of GFA0AdderStr(x, y, z),
 (ii) $y \in$ the carrier of GFA0AdderStr(x, y, z),
 (iii) $z \in$ the carrier of GFA0AdderStr(x, y, z),
 (iv) $\langle\langle x, y\rangle, \mathrm{xor}_2\rangle \in$ the carrier of GFA0AdderStr(x, y, z), and
 (v) $\langle\langle\langle\langle x, y\rangle, \mathrm{xor}_2\rangle, z\rangle, \mathrm{xor}_2\rangle \in$ the carrier of GFA0AdderStr(x, y, z).

(32) For all sets x, y, z holds $\langle\langle x, y\rangle, \mathrm{xor}_2\rangle \in$ InnerVertices(GFA0AdderStr(x, y, z)) and GFA0AdderOutput$(x, y, z) \in$ InnerVertices(GFA0AdderStr(x, y, z)).

(33) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ holds $x \in$ InputVertices(GFA0AdderStr(x, y, z)) and

$y \in$ InputVertices(GFA0AdderStr(x, y, z)) and
$z \in$ InputVertices(GFA0AdderStr(x, y, z)).

(34) For all non pair sets x, y, z holds InputVertices(GFA0AdderStr(x, y, z)) = $\{x, y, z\}$.

(35) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$. Let s be a state of GFA0AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following(s))($\langle\langle x, y\rangle$, $\mathrm{xor}_2\rangle$) $= a_1 \oplus a_2$ and (Following(s))(x) $= a_1$ and (Following(s))(y) $= a_2$ and (Following(s))(z) $= a_3$.

(36) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$. Let s be a state of GFA0AdderCirc(x, y, z) and a_4, a_1, a_2, a_3 be elements of *Boolean*. If $a_4 = s(\langle\langle x, y\rangle, \mathrm{xor}_2\rangle)$ and $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$, then (Following(s))(GFA0AdderOutput(x, y, z)) $= a_4 \oplus a_3$.

(37) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$. Let s be a state of GFA0AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following($s, 2$))(GFA0AdderOutput(x, y, z)) $= a_1 \oplus a_2 \oplus a_3$ and (Following($s, 2$))($\langle\langle x, y\rangle$, $\mathrm{xor}_2\rangle$) $= a_1 \oplus a_2$ and (Following($s, 2$))(x) $= a_1$ and (Following($s, 2$))(y) $= a_2$ and (Following($s, 2$))(z) $= a_3$.

(38) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ and for every state s of GFA0AdderCirc(x, y, z) holds Following($s, 2$) is stable.

Let x, y, z be sets. The functor BitGFA0Str(x, y, z) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined as follows:

(Def. 13) BitGFA0Str(x, y, z) = GFA0AdderStr(x, y, z)+· GFA0CarryStr(x, y, z).

Let x, y, z be sets. The functor BitGFA0Circ(x, y, z) yielding a strict Boolean circuit of BitGFA0Str(x, y, z) with denotation held in gates is defined by:

(Def. 14) BitGFA0Circ(x, y, z) = GFA0AdderCirc(x, y, z)+· GFA0CarryCirc(x, y, z).

We now state several propositions:

(39) For all sets x, y, z holds InnerVertices(BitGFA0Str(x, y, z)) = $\{\langle\langle x, y\rangle, \mathrm{xor}_2\rangle\} \cup \{$GFA0AdderOutput($x, y, z$)$\} \cup \{\langle\langle x, y\rangle, \mathrm{and}_2\rangle, \langle\langle y, z\rangle, \mathrm{and}_2\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\} \cup \{$GFA0CarryOutput($x, y, z$)$\}$.

(40) For all sets x, y, z holds InnerVertices(BitGFA0Str(x, y, z)) is a binary relation.

(41) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_2\rangle$ holds InputVertices(BitGFA0Str(x, y, z)) = $\{x, y, z\}$.

(42) For all non pair sets x, y, z holds InputVertices(BitGFA0Str(x, y, z)) = $\{x, y, z\}$.

(43) For all non pair sets x, y, z holds InputVertices(BitGFA0Str(x, y, z)) has no pairs.

(44) Let x, y, z be sets. Then $x \in$ the carrier of BitGFA0Str(x, y, z) and $y \in$ the carrier of BitGFA0Str(x, y, z) and $z \in$ the carrier of BitGFA0Str(x, y, z) and $\langle\langle x, y\rangle, \mathrm{xor}_2\rangle \in$ the carrier of BitGFA0Str(x, y, z) and $\langle\langle\langle\langle x, y\rangle, \mathrm{xor}_2\rangle, z\rangle, \mathrm{xor}_2\rangle \in$ the carrier of BitGFA0Str(x, y, z) and $\langle\langle x, y\rangle, \mathrm{and}_2\rangle \in$ the carrier of BitGFA0Str(x, y, z) and $\langle\langle y, z\rangle, \mathrm{and}_2\rangle \in$ the carrier of BitGFA0Str(x, y, z) and $\langle\langle z, x\rangle, \mathrm{and}_2\rangle \in$ the carrier of BitGFA0Str(x, y, z) and $\langle\langle\langle\langle x, y\rangle, \mathrm{and}_2\rangle, \langle\langle y, z\rangle, \mathrm{and}_2\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\rangle, \mathrm{or}_3\rangle \in$ the carrier of BitGFA0Str(x, y, z).

(45) Let x, y, z be sets. Then $\langle\langle x, y\rangle, \mathrm{xor}_2\rangle \in$ InnerVertices(BitGFA0Str(x, y, z)) and GFA0AdderOutput$(x, y, z) \in$ InnerVertices(BitGFA0Str(x, y, z)) and $\langle\langle x, y\rangle, \mathrm{and}_2\rangle \in$ InnerVertices(BitGFA0Str(x, y, z)) and $\langle\langle y, z\rangle, \mathrm{and}_2\rangle \in$ InnerVertices(BitGFA0Str(x, y, z)) and $\langle\langle z, x\rangle, \mathrm{and}_2\rangle \in$ InnerVertices(BitGFA0Str(x, y, z)) and GFA0CarryOutput$(x, y, z) \in$ InnerVertices(BitGFA0Str(x, y, z)).

(46) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_2\rangle$. Then $x \in$ InputVertices(BitGFA0Str(x, y, z)) and $y \in$ InputVertices(BitGFA0Str(x, y, z)) and $z \in$ InputVertices(BitGFA0Str(x, y, z)).

Let x, y, z be sets. The functor BitGFA0CarryOutput(x, y, z) yielding an element of InnerVertices(BitGFA0Str(x, y, z)) is defined as follows:

(Def. 15) BitGFA0CarryOutput$(x, y, z) = \langle\langle\langle\langle x, y\rangle, \mathrm{and}_2\rangle, \langle\langle y, z\rangle, \mathrm{and}_2\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\rangle, \mathrm{or}_3\rangle$.

Let x, y, z be sets. The functor BitGFA0AdderOutput(x, y, z) yielding an element of InnerVertices(BitGFA0Str(x, y, z)) is defined as follows:

(Def. 16) BitGFA0AdderOutput$(x, y, z) =$ 2GatesCircOutput$(x, y, z, \mathrm{xor}_2)$.

One can prove the following two propositions:

(47) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_2\rangle$. Let s be a state of BitGFA0Circ(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA0AdderOutput(x, y, z)) $= a_1 \oplus a_2 \oplus a_3$ and (Following$(s, 2)$)(GFA0CarryOutput(x, y, z)) $= a_1 \wedge a_2 \vee a_2 \wedge a_3 \vee a_3 \wedge a_1$.

(48) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor}_2\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and}_2\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_2\rangle$. Let s be a state of BitGFA0Circ(x, y, z). Then Following$(s, 2)$ is stable.

3. Generalized Full Adder (GFA) Circuit (TYPE-1)

Let x, y, z be sets. The functor $\mathrm{GFA1CarryIStr}(x, y, z)$ yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

(Def. 17) $\mathrm{GFA1CarryIStr}(x, y, z) = \mathrm{1GateCircStr}(\langle x, y\rangle, \mathrm{and2c}) +\cdot \mathrm{1GateCircStr}(\langle y, z\rangle, \mathrm{and}_{2a}) +\cdot \mathrm{1GateCircStr}(\langle z, x\rangle, \mathrm{and}_2)$.

Let x, y, z be sets. The functor $\mathrm{GFA1CarryICirc}(x, y, z)$ yields a strict Boolean circuit of $\mathrm{GFA1CarryIStr}(x, y, z)$ with denotation held in gates and is defined as follows:

(Def. 18) $\mathrm{GFA1CarryICirc}(x, y, z) = \mathrm{1GateCircuit}(x, y, \mathrm{and2c}) +\cdot \mathrm{1GateCircuit}(y, z, \mathrm{and}_{2a}) +\cdot \mathrm{1GateCircuit}(z, x, \mathrm{and}_2)$.

Let x, y, z be sets. The functor $\mathrm{GFA1CarryStr}(x, y, z)$ yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

(Def. 19) $\mathrm{GFA1CarryStr}(x, y, z) = \mathrm{GFA1CarryIStr}(x, y, z) +\cdot \mathrm{1GateCircStr}(\langle\langle\langle x, y\rangle, \mathrm{and2c}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\rangle, \mathrm{or}_3)$.

Let x, y, z be sets. The functor $\mathrm{GFA1CarryCirc}(x, y, z)$ yielding a strict Boolean circuit of $\mathrm{GFA1CarryStr}(x, y, z)$ with denotation held in gates is defined by:

(Def. 20) $\mathrm{GFA1CarryCirc}(x, y, z) = \mathrm{GFA1CarryICirc}(x, y, z) +\cdot \mathrm{1GateCircuit}(\langle\langle x, y\rangle, \mathrm{and2c}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle, \mathrm{or}_3)$.

Let x, y, z be sets. The functor $\mathrm{GFA1CarryOutput}(x, y, z)$ yielding an element of $\mathrm{InnerVertices}(\mathrm{GFA1CarryStr}(x, y, z))$ is defined as follows:

(Def. 21) $\mathrm{GFA1CarryOutput}(x, y, z) = \langle\langle\langle\langle x, y\rangle, \mathrm{and2c}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\rangle, \mathrm{or}_3\rangle$.

We now state a number of propositions:

(49) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{GFA1CarryIStr}(x, y, z)) = \{\langle\langle x, y\rangle, \mathrm{and2c}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\}$.

(50) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{GFA1CarryStr}(x, y, z)) = \{\langle\langle x, y\rangle, \mathrm{and2c}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\} \cup \{\mathrm{GFA1CarryOutput}(x, y, z)\}$.

(51) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{GFA1CarryStr}(x, y, z))$ is a binary relation.

(52) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and2c}\rangle$ holds $\mathrm{InputVertices}(\mathrm{GFA1CarryIStr}(x, y, z)) = \{x, y, z\}$.

(53) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and2c}\rangle$ holds $\mathrm{InputVertices}(\mathrm{GFA1CarryStr}(x, y, z)) = \{x, y, z\}$.

(54) For all non pair sets x, y, z holds InputVertices(GFA1CarryStr(x, y, z)) has no pairs.

(55) Let x, y, z be sets. Then $x \in$ the carrier of GFA1CarryStr(x, y, z) and $y \in$ the carrier of GFA1CarryStr(x, y, z) and $z \in$ the carrier of GFA1CarryStr(x, y, z) and $\langle\langle x, y\rangle, \text{and2c}\rangle \in$ the carrier of GFA1CarryStr(x, y, z) and $\langle\langle y, z\rangle, \text{and}_{2a}\rangle \in$ the carrier of GFA1CarryStr(x, y, z) and $\langle\langle z, x\rangle, \text{and}_2\rangle \in$ the carrier of GFA1CarryStr(x, y, z) and $\langle\langle\langle\langle x, y\rangle, \text{and2c}\rangle, \langle\langle y, z\rangle, \text{and}_{2a}\rangle, \langle\langle z, x\rangle, \text{and}_2\rangle\rangle, \text{or}_3\rangle \in$ the carrier of GFA1CarryStr(x, y, z).

(56) For all sets x, y, z holds $\langle\langle x, y\rangle, \text{and2c}\rangle \in$ InnerVertices(GFA1CarryStr(x, y, z)) and $\langle\langle y, z\rangle, \text{and}_{2a}\rangle \in$ InnerVertices(GFA1CarryStr(x, y, z)) and $\langle\langle z, x\rangle, \text{and}_2\rangle \in$ InnerVertices(GFA1CarryStr(x, y, z)) and GFA1CarryOutput$(x, y, z) \in$ InnerVertices(GFA1CarryStr(x, y, z)).

(57) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \text{and}_{2a}\rangle$ and $y \neq \langle\langle z, x\rangle, \text{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \text{and2c}\rangle$ holds $x \in$ InputVertices(GFA1CarryStr(x, y, z)) and $y \in$ InputVertices(GFA1CarryStr(x, y, z)) and $z \in$ InputVertices(GFA1CarryStr(x, y, z)).

(58) For all non pair sets x, y, z holds InputVertices(GFA1CarryStr(x, y, z)) $= \{x, y, z\}$.

(59) Let x, y, z be sets, s be a state of GFA1CarryCirc(x, y, z), and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following(s))$(\langle\langle x, y\rangle, \text{and2c}\rangle) = a_1 \wedge \neg a_2$ and (Following(s))$(\langle\langle y, z\rangle, \text{and}_{2a}\rangle) = \neg a_2 \wedge a_3$ and (Following(s))$(\langle\langle z, x\rangle, \text{and}_2\rangle) = a_3 \wedge a_1$.

(60) Let x, y, z be sets, s be a state of GFA1CarryCirc(x, y, z), and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(\langle\langle x, y\rangle, \text{and2c}\rangle)$ and $a_2 = s(\langle\langle y, z\rangle, \text{and}_{2a}\rangle)$ and $a_3 = s(\langle\langle z, x\rangle, \text{and}_2\rangle)$, then (Following$(s)$)(GFA1CarryOutput$(x, y, z)$) $= a_1 \vee a_2 \vee a_3$.

(61) Let x, y, z be sets. Suppose $x \neq \langle\langle y, z\rangle, \text{and}_{2a}\rangle$ and $y \neq \langle\langle z, x\rangle, \text{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \text{and2c}\rangle$. Let s be a state of GFA1CarryCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA1CarryOutput(x, y, z)) $= a_1 \wedge \neg a_2 \vee \neg a_2 \wedge a_3 \vee a_3 \wedge a_1$ and (Following$(s, 2)$)$(\langle\langle x, y\rangle, \text{and2c}\rangle) = a_1 \wedge \neg a_2$ and (Following$(s, 2)$)$(\langle\langle y, z\rangle, \text{and}_{2a}\rangle) = \neg a_2 \wedge a_3$ and (Following$(s, 2)$)$(\langle\langle z, x\rangle, \text{and}_2\rangle) = a_3 \wedge a_1$.

(62) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \text{and}_{2a}\rangle$ and $y \neq \langle\langle z, x\rangle, \text{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \text{and2c}\rangle$ and for every state s of GFA1CarryCirc(x, y, z) holds Following$(s, 2)$ is stable.

Let x, y, z be sets. The functor GFA1AdderStr(x, y, z) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined as follows:

(Def. 22) $\text{GFA1AdderStr}(x, y, z) = \text{2GatesCircStr}(x, y, z, \text{xor2c})$.

Let x, y, z be sets. The functor $\text{GFA1AdderCirc}(x, y, z)$ yielding a strict Boolean circuit of $\text{GFA1AdderStr}(x, y, z)$ with denotation held in gates is defined by:

(Def. 23) $\text{GFA1AdderCirc}(x, y, z) = \text{2GatesCircuit}(x, y, z, \text{xor2c})$.

Let x, y, z be sets. The functor $\text{GFA1AdderOutput}(x, y, z)$ yields an element of $\text{InnerVertices}(\text{GFA1AdderStr}(x, y, z))$ and is defined as follows:

(Def. 24) $\text{GFA1AdderOutput}(x, y, z) = \text{2GatesCircOutput}(x, y, z, \text{xor2c})$.

We now state a number of propositions:

(63) For all sets x, y, z holds $\text{InnerVertices}(\text{GFA1AdderStr}(x, y, z)) = \{\langle\langle x, y\rangle, \text{xor2c}\rangle\} \cup \{\text{GFA1AdderOutput}(x, y, z)\}$.

(64) For all sets x, y, z holds $\text{InnerVertices}(\text{GFA1AdderStr}(x, y, z))$ is a binary relation.

(65) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \text{xor2c}\rangle$ holds $\text{InputVertices}(\text{GFA1AdderStr}(x, y, z)) = \{x, y, z\}$.

(66) For all non pair sets x, y, z holds $\text{InputVertices}(\text{GFA1AdderStr}(x, y, z))$ has no pairs.

(67) Let x, y, z be sets. Then

(i) $x \in$ the carrier of $\text{GFA1AdderStr}(x, y, z)$,

(ii) $y \in$ the carrier of $\text{GFA1AdderStr}(x, y, z)$,

(iii) $z \in$ the carrier of $\text{GFA1AdderStr}(x, y, z)$,

(iv) $\langle\langle x, y\rangle, \text{xor2c}\rangle \in$ the carrier of $\text{GFA1AdderStr}(x, y, z)$, and

(v) $\langle\langle\langle\langle x, y\rangle, \text{xor2c}\rangle, z\rangle, \text{xor2c}\rangle \in$ the carrier of $\text{GFA1AdderStr}(x, y, z)$.

(68) For all sets x, y, z holds $\langle\langle x, y\rangle, \text{xor2c}\rangle \in \text{InnerVertices}(\text{GFA1AdderStr}(x, y, z))$ and $\text{GFA1AdderOutput}(x, y, z) \in \text{InnerVertices}(\text{GFA1AdderStr}(x, y, z))$.

(69) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \text{xor2c}\rangle$ holds $x \in \text{InputVertices}(\text{GFA1AdderStr}(x, y, z))$ and
$y \in \text{InputVertices}(\text{GFA1AdderStr}(x, y, z))$ and
$z \in \text{InputVertices}(\text{GFA1AdderStr}(x, y, z))$.

(70) For all non pair sets x, y, z holds $\text{InputVertices}(\text{GFA1AdderStr}(x, y, z)) = \{x, y, z\}$.

(71) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \text{xor2c}\rangle$. Let s be a state of $\text{GFA1AdderCirc}(x, y, z)$ and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then $(\text{Following}(s))(\langle\langle x, y\rangle, \text{xor2c}\rangle) = a_1 \oplus \neg a_2$ and $(\text{Following}(s))(x) = a_1$ and $(\text{Following}(s))(y) = a_2$ and $(\text{Following}(s))(z) = a_3$.

(72) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \text{xor2c}\rangle$. Let s be a state of $\text{GFA1AdderCirc}(x, y, z)$ and a_4, a_1, a_2, a_3 be elements of *Boolean*. If

$a_4 = s(\langle\langle x, y\rangle, \mathrm{xor2c}\rangle)$ and $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$, then $(\mathrm{Following}(s))(\mathrm{GFA1AdderOutput}(x, y, z)) = a_4 \oplus \neg a_3$.

(73) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$. Let s be a state of $\mathrm{GFA1AdderCirc}(x, y, z)$ and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then $(\mathrm{Following}(s, 2))(\mathrm{GFA1AdderOutput}(x, y, z)) = a_1 \oplus \neg a_2 \oplus \neg a_3$ and $(\mathrm{Following}(s, 2))(\langle\langle x, y\rangle, \mathrm{xor2c}\rangle) = a_1 \oplus \neg a_2$ and $(\mathrm{Following}(s, 2))(x) = a_1$ and $(\mathrm{Following}(s, 2))(y) = a_2$ and $(\mathrm{Following}(s, 2))(z) = a_3$.

(74) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$. Let s be a state of $\mathrm{GFA1AdderCirc}(x, y, z)$ and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$, then $(\mathrm{Following}(s, 2))(\mathrm{GFA1AdderOutput}(x, y, z)) = \neg(a_1 \oplus \neg a_2 \oplus a_3)$.

(75) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ and for every state s of $\mathrm{GFA1AdderCirc}(x, y, z)$ holds $\mathrm{Following}(s, 2)$ is stable.

Let x, y, z be sets. The functor $\mathrm{BitGFA1Str}(x, y, z)$ yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined as follows:

(Def. 25) $\mathrm{BitGFA1Str}(x, y, z) = \mathrm{GFA1AdderStr}(x, y, z) +\cdot \mathrm{GFA1CarryStr}(x, y, z)$.

Let x, y, z be sets. The functor $\mathrm{BitGFA1Circ}(x, y, z)$ yielding a strict Boolean circuit of $\mathrm{BitGFA1Str}(x, y, z)$ with denotation held in gates is defined by:

(Def. 26) $\mathrm{BitGFA1Circ}(x, y, z) = \mathrm{GFA1AdderCirc}(x, y, z) +\cdot \mathrm{GFA1CarryCirc}(x, y, z)$.

We now state several propositions:

(76) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{BitGFA1Str}(x, y, z)) = \{\langle\langle x, y\rangle, \mathrm{xor2c}\rangle\} \cup \{\mathrm{GFA1AdderOutput}(x, y, z)\} \cup \{\langle\langle x, y\rangle, \mathrm{and2c}\rangle, \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle, \langle\langle z, x\rangle, \mathrm{and}_2\rangle\} \cup \{\mathrm{GFA1CarryOutput}(x, y, z)\}$.

(77) For all sets x, y, z holds $\mathrm{InnerVertices}(\mathrm{BitGFA1Str}(x, y, z))$ is a binary relation.

(78) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and}_{2a}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and2c}\rangle$ holds $\mathrm{InputVertices}(\mathrm{BitGFA1Str}(x, y, z)) = \{x, y, z\}$.

(79) For all non pair sets x, y, z holds $\mathrm{InputVertices}(\mathrm{BitGFA1Str}(x, y, z)) = \{x, y, z\}$.

(80) For all non pair sets x, y, z holds $\mathrm{InputVertices}(\mathrm{BitGFA1Str}(x, y, z))$ has no pairs.

(81) Let x, y, z be sets. Then $x \in$ the carrier of $\mathrm{BitGFA1Str}(x, y, z)$ and $y \in$ the carrier of $\mathrm{BitGFA1Str}(x, y, z)$ and $z \in$ the carrier of $\mathrm{BitGFA1Str}(x, y, z)$ and $\langle\langle x, y\rangle, \mathrm{xor2c}\rangle \in$ the carrier of $\mathrm{BitGFA1Str}(x, y, z)$ and $\langle\langle\langle\langle x, y\rangle, \mathrm{xor2c}\rangle, z\rangle, \mathrm{xor2c}\rangle \in$ the carrier of $\mathrm{BitGFA1Str}(x, y, z)$ and $\langle\langle x, y\rangle, \mathrm{and2c}\rangle \in$ the car-

rier of BitGFA1Str(x, y, z) and $\langle\langle y, z\rangle, \text{and}_{2a}\rangle \in$ the carrier of BitGFA1Str(x, y, z) and $\langle\langle z, x\rangle, \text{and}_2\rangle \in$ the carrier of BitGFA1Str(x, y, z) and $\langle\langle\langle\langle x, y\rangle, \text{and2c}\rangle, \langle\langle y, z\rangle, \text{and}_{2a}\rangle, \langle\langle z, x\rangle, \text{and}_2\rangle\rangle, \text{or}_3\rangle \in$ the carrier of BitGFA1Str(x, y, z).

(82) Let x, y, z be sets. Then $\langle\langle x, y\rangle, \text{xor2c}\rangle \in$ InnerVertices(BitGFA1Str(x, y, z)) and GFA1AdderOutput$(x, y, z) \in$ InnerVertices(BitGFA1Str(x, y, z)) and $\langle\langle x, y\rangle, \text{and2c}\rangle \in$ InnerVertices(BitGFA1Str(x, y, z)) and $\langle\langle y, z\rangle, \text{and}_{2a}\rangle \in$ InnerVertices(BitGFA1Str(x, y, z)) and $\langle\langle z, x\rangle, \text{and}_2\rangle \in$ InnerVertices(BitGFA1Str(x, y, z)) and GFA1CarryOutput$(x, y, z) \in$ InnerVertices(BitGFA1Str(x, y, z)).

(83) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \text{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \text{and}_{2a}\rangle$ and $y \neq \langle\langle z, x\rangle, \text{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \text{and2c}\rangle$. Then $x \in$ InputVertices(BitGFA1Str(x, y, z)) and $y \in$ InputVertices(BitGFA1Str(x, y, z)) and $z \in$ InputVertices(BitGFA1Str(x, y, z)).

Let x, y, z be sets. The functor BitGFA1CarryOutput(x, y, z) yielding an element of InnerVertices(BitGFA1Str(x, y, z)) is defined as follows:

(Def. 27) BitGFA1CarryOutput$(x, y, z) = \langle\langle\langle\langle x, y\rangle, \text{and2c}\rangle, \langle\langle y, z\rangle, \text{and}_{2a}\rangle, \langle\langle z, x\rangle, \text{and}_2\rangle\rangle, \text{or}_3\rangle$.

Let x, y, z be sets. The functor BitGFA1AdderOutput(x, y, z) yielding an element of InnerVertices(BitGFA1Str(x, y, z)) is defined as follows:

(Def. 28) BitGFA1AdderOutput$(x, y, z) =$ 2GatesCircOutput$(x, y, z, \text{xor2c})$.

The following two propositions are true:

(84) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \text{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \text{and}_{2a}\rangle$ and $y \neq \langle\langle z, x\rangle, \text{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \text{and2c}\rangle$. Let s be a state of BitGFA1Circ(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA1AdderOutput(x, y, z)) $= \neg(a_1 \oplus \neg a_2 \oplus a_3)$ and (Following$(s, 2)$)(GFA1CarryOutput(x, y, z)) $= a_1 \wedge \neg a_2 \vee \neg a_2 \wedge a_3 \vee a_3 \wedge a_1$.

(85) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \text{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \text{and}_{2a}\rangle$ and $y \neq \langle\langle z, x\rangle, \text{and}_2\rangle$ and $z \neq \langle\langle x, y\rangle, \text{and2c}\rangle$. Let s be a state of BitGFA1Circ(x, y, z). Then Following$(s, 2)$ is stable.

4. Generalized Full Adder (GFA) Circuit (TYPE-2)

Let x, y, z be sets. The functor GFA2CarryIStr(x, y, z) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined by:

(Def. 29) GFA2CarryIStr$(x, y, z) =$ 1GateCircStr$(\langle x, y\rangle, \text{and}_{2a})$+· 1GateCircStr$(\langle y, z\rangle, \text{and2c})$+· 1GateCircStr$(\langle z, x\rangle, \text{and}_{2b})$.

Let x, y, z be sets. The functor GFA2CarryICirc(x, y, z) yielding a strict Boolean circuit of GFA2CarryIStr(x, y, z) with denotation held in gates is defined as follows:

(Def. 30) GFA2CarryICirc$(x, y, z) =$ 1GateCircuit$(x, y, \mathrm{and}_{2a})$+· 1GateCircuit$(y, z, \mathrm{and2c})$+· 1GateCircuit$(z, x, \mathrm{and}_{2b})$.

Let x, y, z be sets. The functor GFA2CarryStr(x, y, z) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined as follows:

(Def. 31) GFA2CarryStr$(x, y, z) =$ GFA2CarryIStr(x, y, z)+· 1GateCircStr$(\langle\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, z\rangle, \mathrm{and2c}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\rangle, \mathrm{nor}_3)$.

Let x, y, z be sets. The functor GFA2CarryCirc(x, y, z) yields a strict Boolean circuit of GFA2CarryStr(x, y, z) with denotation held in gates and is defined as follows:

(Def. 32) GFA2CarryCirc$(x, y, z) =$ GFA2CarryICirc(x, y, z)+· 1GateCircuit$(\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, z\rangle, \mathrm{and2c}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle, \mathrm{nor}_3)$.

Let x, y, z be sets. The functor GFA2CarryOutput(x, y, z) yields an element of InnerVertices(GFA2CarryStr(x, y, z)) and is defined by:

(Def. 33) GFA2CarryOutput$(x, y, z) = \langle\langle\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, z\rangle, \mathrm{and2c}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\rangle, \mathrm{nor}_3\rangle$.

We now state a number of propositions:

(86) For all sets x, y, z holds InnerVertices(GFA2CarryIStr(x, y, z)) $= \{\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, z\rangle, \mathrm{and2c}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\}$.

(87) For all sets x, y, z holds InnerVertices(GFA2CarryStr(x, y, z)) $= \{\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, z\rangle, \mathrm{and2c}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\} \cup \{$GFA2CarryOutput$(x, y, z)\}$.

(88) For all sets x, y, z holds InnerVertices(GFA2CarryStr(x, y, z)) is a binary relation.

(89) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ holds InputVertices(GFA2CarryIStr(x, y, z)) $= \{x, y, z\}$.

(90) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ holds InputVertices(GFA2CarryStr(x, y, z)) $= \{x, y, z\}$.

(91) For all non pair sets x, y, z holds InputVertices(GFA2CarryStr(x, y, z)) has no pairs.

(92) Let x, y, z be sets. Then $x \in$ the carrier of GFA2CarryStr(x, y, z) and $y \in$ the carrier of GFA2CarryStr(x, y, z) and $z \in$ the carrier of GFA2CarryStr(x, y, z) and $\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle \in$ the carrier of GFA2CarryStr(x, y, z) and $\langle\langle y, z\rangle, \mathrm{and2c}\rangle \in$ the carrier of GFA2CarryStr(x, y, z) and $\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle \in$ the carrier

of GFA2CarryStr(x, y, z) and $\langle\langle\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, z\rangle, \mathrm{and2c}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\rangle, \mathrm{nor}_3\rangle \in$ the carrier of GFA2CarryStr(x, y, z).

(93) For all sets x, y, z holds $\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle \in$ InnerVertices(GFA2CarryStr(x, y, z)) and $\langle\langle y, z\rangle, \mathrm{and2c}\rangle \in$ InnerVertices(GFA2CarryStr(x, y, z)) and $\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle \in$ InnerVertices(GFA2CarryStr(x, y, z)) and GFA2CarryOutput $(x, y, z) \in$ InnerVertices(GFA2CarryStr(x, y, z)).

(94) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ holds $x \in$ InputVertices(GFA2CarryStr(x, y, z)) and $y \in$ InputVertices(GFA2CarryStr(x, y, z)) and $z \in$ InputVertices(GFA2CarryStr(x, y, z)).

(95) For all non pair sets x, y, z holds InputVertices(GFA2CarryStr(x, y, z)) = $\{x, y, z\}$.

(96) Let x, y, z be sets, s be a state of GFA2CarryCirc(x, y, z), and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following(s))($\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$) $= \neg a_1 \wedge a_2$ and (Following(s))($\langle\langle y, z\rangle, \mathrm{and2c}\rangle$) $= a_2 \wedge \neg a_3$ and (Following(s))($\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$) $= \neg a_3 \wedge \neg a_1$.

(97) Let x, y, z be sets, s be a state of GFA2CarryCirc(x, y, z), and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle)$ and $a_2 = s(\langle\langle y, z\rangle, \mathrm{and2c}\rangle)$ and $a_3 = s(\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle)$, then (Following$(s)$)(GFA2CarryOutput$(x, y, z)$) $= \neg(a_1 \vee a_2 \vee a_3)$.

(98) Let x, y, z be sets. Suppose $x \neq \langle\langle y, z\rangle, \mathrm{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$. Let s be a state of GFA2CarryCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA2CarryOutput(x, y, z)) $= \neg(\neg a_1 \wedge a_2 \vee a_2 \wedge \neg a_3 \vee \neg a_3 \wedge \neg a_1)$ and (Following$(s, 2)$)($\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$) $= \neg a_1 \wedge a_2$ and (Following$(s, 2)$)($\langle\langle y, z\rangle, \mathrm{and2c}\rangle$) $= a_2 \wedge \neg a_3$ and (Following$(s, 2)$)($\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$) $= \neg a_3 \wedge \neg a_1$.

(99) For all sets x, y, z such that $x \neq \langle\langle y, z\rangle, \mathrm{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$ and for every state s of GFA2CarryCirc(x, y, z) holds Following$(s, 2)$ is stable.

Let x, y, z be sets. The functor GFA2AdderStr(x, y, z) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined as follows:

(Def. 34) GFA2AdderStr(x, y, z) = 2GatesCircStr$(x, y, z, \mathrm{xor2c})$.

Let x, y, z be sets. The functor GFA2AdderCirc(x, y, z) yielding a strict Boolean circuit of GFA2AdderStr(x, y, z) with denotation held in gates is defined as follows:

(Def. 35) GFA2AdderCirc(x, y, z) = 2GatesCircuit$(x, y, z, \mathrm{xor2c})$.

Let x, y, z be sets. The functor GFA2AdderOutput(x, y, z) yields an element of InnerVertices(GFA2AdderStr(x, y, z)) and is defined by:

(Def. 36) GFA2AdderOutput(x, y, z) = 2GatesCircOutput$(x, y, z, \mathrm{xor2c})$.

One can prove the following propositions:

(100) For all sets x, y, z holds InnerVertices(GFA2AdderStr(x, y, z)) = $\{\langle\langle x, y\rangle, \mathrm{xor2c}\rangle\} \cup \{$GFA2AdderOutput$(x, y, z)\}$.

(101) For all sets x, y, z holds InnerVertices(GFA2AdderStr(x, y, z)) is a binary relation.

(102) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ holds InputVertices(GFA2AdderStr(x, y, z)) = $\{x, y, z\}$.

(103) For all non pair sets x, y, z holds InputVertices(GFA2AdderStr(x, y, z)) has no pairs.

(104) Let x, y, z be sets. Then

(i) $x \in$ the carrier of GFA2AdderStr(x, y, z),

(ii) $y \in$ the carrier of GFA2AdderStr(x, y, z),

(iii) $z \in$ the carrier of GFA2AdderStr(x, y, z),

(iv) $\langle\langle x, y\rangle, \mathrm{xor2c}\rangle \in$ the carrier of GFA2AdderStr(x, y, z), and

(v) $\langle\langle\langle\langle x, y\rangle, \mathrm{xor2c}\rangle, z\rangle, \mathrm{xor2c}\rangle \in$ the carrier of GFA2AdderStr(x, y, z).

(105) For all sets x, y, z holds $\langle\langle x, y\rangle, \mathrm{xor2c}\rangle \in$ InnerVertices(GFA2AdderStr(x, y, z)) and GFA2AdderOutput$(x, y, z) \in$ InnerVertices(GFA2AdderStr(x, y, z)).

(106) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ holds $x \in$ InputVertices(GFA2AdderStr(x, y, z)) and
$y \in$ InputVertices(GFA2AdderStr(x, y, z)) and
$z \in$ InputVertices(GFA2AdderStr(x, y, z)).

(107) For all non pair sets x, y, z holds InputVertices(GFA2AdderStr(x, y, z)) = $\{x, y, z\}$.

(108) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$. Let s be a state of GFA2AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following(s))$(\langle\langle x, y\rangle, \mathrm{xor2c}\rangle) = a_1 \oplus \neg a_2$ and (Following(s))$(x) = a_1$ and (Following(s))$(y) = a_2$ and (Following(s))$(z) = a_3$.

(109) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$. Let s be a state of GFA2AdderCirc(x, y, z) and a_4, a_1, a_2, a_3 be elements of *Boolean*. If $a_4 = s(\langle\langle x, y\rangle, \mathrm{xor2c}\rangle)$ and $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$, then (Following(s))(GFA2AdderOutput(x, y, z)) $= a_4 \oplus \neg a_3$.

(110) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$. Let s be a state of GFA2AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA2AdderOutput(x, y, z)) $= a_1 \oplus \neg a_2 \oplus \neg a_3$ and (Following$(s, 2)$)$(\langle\langle x, y\rangle, \mathrm{xor2c}\rangle) = a_1 \oplus \neg a_2$ and (Following$(s, 2)$)$(x) = a_1$ and (Following$(s, 2)$)$(y) = a_2$ and (Following$(s, 2)$)$(z) = a_3$.

(111) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \text{xor2c}\rangle$. Let s be a state of GFA2AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$, then (Following$(s, 2)$)(GFA2AdderOutput(x, y, z)) $= \neg a_1 \oplus a_2 \oplus \neg a_3$.

(112) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \text{xor2c}\rangle$ and for every state s of GFA2AdderCirc(x, y, z) holds Following$(s, 2)$ is stable.

Let x, y, z be sets. The functor BitGFA2Str(x, y, z) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined as follows:

(Def. 37) BitGFA2Str(x, y, z) = GFA2AdderStr(x, y, z)+· GFA2CarryStr(x, y, z).

Let x, y, z be sets. The functor BitGFA2Circ(x, y, z) yields a strict Boolean circuit of BitGFA2Str(x, y, z) with denotation held in gates and is defined by:

(Def. 38) BitGFA2Circ(x, y, z) = GFA2AdderCirc(x, y, z)+· GFA2CarryCirc(x, y, z).

Next we state several propositions:

(113) For all sets x, y, z holds InnerVertices(BitGFA2Str(x, y, z)) $= \{\langle\langle x, y\rangle, \text{xor2c}\rangle\} \cup \{\text{GFA2AdderOutput}(x, y, z)\} \cup \{\langle\langle x, y\rangle, \text{and}_{2a}\rangle, \langle\langle y, z\rangle, \text{and2c}\rangle, \langle\langle z, x\rangle, \text{and}_{2b}\rangle\} \cup \{\text{GFA2CarryOutput}(x, y, z)\}$.

(114) For all sets x, y, z holds InnerVertices(BitGFA2Str(x, y, z)) is a binary relation.

(115) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \text{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \text{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \text{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \text{and}_{2a}\rangle$ holds InputVertices(BitGFA2Str(x, y, z)) $= \{x, y, z\}$.

(116) For all non pair sets x, y, z holds InputVertices(BitGFA2Str(x, y, z)) $= \{x, y, z\}$.

(117) For all non pair sets x, y, z holds InputVertices(BitGFA2Str(x, y, z)) has no pairs.

(118) Let x, y, z be sets. Then $x \in$ the carrier of BitGFA2Str(x, y, z) and $y \in$ the carrier of BitGFA2Str(x, y, z) and $z \in$ the carrier of BitGFA2Str(x, y, z) and $\langle\langle x, y\rangle, \text{xor2c}\rangle \in$ the carrier of BitGFA2Str(x, y, z) and $\langle\langle\langle\langle x, y\rangle, \text{xor2c}\rangle, z\rangle, \text{xor2c}\rangle \in$ the carrier of BitGFA2Str(x, y, z) and $\langle\langle x, y\rangle, \text{and}_{2a}\rangle \in$ the carrier of BitGFA2Str(x, y, z) and $\langle\langle y, z\rangle, \text{and2c}\rangle \in$ the carrier of BitGFA2Str(x, y, z) and $\langle\langle z, x\rangle, \text{and}_{2b}\rangle \in$ the carrier of BitGFA2Str(x, y, z) and $\langle\langle\langle\langle x, y\rangle, \text{and}_{2a}\rangle, \langle\langle y, z\rangle, \text{and2c}\rangle, \langle\langle z, x\rangle, \text{and}_{2b}\rangle\rangle, \text{nor}_3\rangle \in$ the carrier of BitGFA2Str(x, y, z).

(119) Let x, y, z be sets. Then $\langle\langle x, y\rangle, \text{xor2c}\rangle \in$ InnerVertices(BitGFA2Str(x, y, z)) and GFA2AdderOutput$(x, y, z) \in$ InnerVertices(BitGFA2Str(x, y, z)) and $\langle\langle x, y\rangle, \text{and}_{2a}\rangle \in$ InnerVertices(BitGFA2Str(x, y, z)) and $\langle\langle y, z\rangle, \text{and2c}\rangle \in$ InnerVertices(BitGFA2Str(x, y, z)) and $\langle\langle z, x\rangle, \text{and}_{2b}\rangle \in$ InnerVertices(BitGFA2Str(x, y, z)) and GFA2CarryOutput$(x, y, z) \in$

InnerVertices(BitGFA2Str(x, y, z)).

(120) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$. Then $x \in$ InputVertices(BitGFA2Str(x, y, z)) and $y \in$ InputVertices(BitGFA2Str(x, y, z)) and $z \in$ InputVertices(BitGFA2Str(x, y, z)).

Let x, y, z be sets. The functor BitGFA2CarryOutput(x, y, z) yields an element of InnerVertices(BitGFA2Str(x, y, z)) and is defined by:

(Def. 39) BitGFA2CarryOutput$(x, y, z) = \langle\langle\langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle, \langle\langle y, z\rangle, \mathrm{and2c}\rangle, \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\rangle, \mathrm{nor}_3\rangle$.

Let x, y, z be sets. The functor BitGFA2AdderOutput(x, y, z) yielding an element of InnerVertices(BitGFA2Str(x, y, z)) is defined by:

(Def. 40) BitGFA2AdderOutput$(x, y, z) =$ 2GatesCircOutput$(x, y, z, \mathrm{xor2c})$.

Next we state two propositions:

(121) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$. Let s be a state of BitGFA2Circ(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA2AdderOutput(x, y, z)) $= \neg a_1 \oplus a_2 \oplus \neg a_3$ and (Following$(s, 2)$)(GFA2CarryOutput(x, y, z)) $= \neg(\neg a_1 \wedge a_2 \vee a_2 \wedge \neg a_3 \vee \neg a_3 \wedge \neg a_1)$.

(122) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \mathrm{xor2c}\rangle$ and $x \neq \langle\langle y, z\rangle, \mathrm{and2c}\rangle$ and $y \neq \langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \mathrm{and}_{2a}\rangle$. Let s be a state of BitGFA2Circ(x, y, z). Then Following$(s, 2)$ is stable.

5. Generalized Full Adder (GFA) Circuit (TYPE-3)

Let x, y, z be sets. The functor GFA3CarryIStr(x, y, z) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

(Def. 41) GFA3CarryIStr$(x, y, z) =$ 1GateCircStr$(\langle x, y\rangle, \mathrm{and}_{2b}) +\cdot$ 1GateCircStr$(\langle y, z\rangle, \mathrm{and}_{2b}) +\cdot$ 1GateCircStr$(\langle z, x\rangle, \mathrm{and}_{2b})$.

Let x, y, z be sets. The functor GFA3CarryICirc(x, y, z) yielding a strict Boolean circuit of GFA3CarryIStr(x, y, z) with denotation held in gates is defined by:

(Def. 42) GFA3CarryICirc$(x, y, z) =$ 1GateCircuit$(x, y, \mathrm{and}_{2b}) +\cdot$ 1GateCircuit$(y, z, \mathrm{and}_{2b}) +\cdot$ 1GateCircuit$(z, x, \mathrm{and}_{2b})$.

Let x, y, z be sets. The functor GFA3CarryStr(x, y, z) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates is defined by:

(Def. 43) $\text{GFA3CarryStr}(x,y,z) = \text{GFA3CarryIStr}(x,y,z) +\cdot \text{1GateCircStr}(\langle\langle\langle x, y\rangle, \text{and}_{2b}\rangle, \langle\langle y,z\rangle, \text{and}_{2b}\rangle, \langle\langle z,x\rangle, \text{and}_{2b}\rangle\rangle, \text{nor}_3)$.

Let x, y, z be sets. The functor $\text{GFA3CarryCirc}(x,y,z)$ yielding a strict Boolean circuit of $\text{GFA3CarryStr}(x,y,z)$ with denotation held in gates is defined by:

(Def. 44) $\text{GFA3CarryCirc}(x,y,z) = \text{GFA3CarryICirc}(x,y,z) +\cdot \text{1GateCircuit}(\langle\langle x, y\rangle, \text{and}_{2b}\rangle, \langle\langle y,z\rangle, \text{and}_{2b}\rangle, \langle\langle z,x\rangle, \text{and}_{2b}\rangle, \text{nor}_3)$.

Let x, y, z be sets. The functor $\text{GFA3CarryOutput}(x,y,z)$ yields an element of $\text{InnerVertices}(\text{GFA3CarryStr}(x,y,z))$ and is defined as follows:

(Def. 45) $\text{GFA3CarryOutput}(x,y,z) = \langle\langle\langle\langle x,y\rangle, \text{and}_{2b}\rangle, \langle\langle y,z\rangle, \text{and}_{2b}\rangle, \langle\langle z,x\rangle, \text{and}_{2b}\rangle\rangle, \text{nor}_3\rangle$.

The following propositions are true:

(123) For all sets x, y, z holds $\text{InnerVertices}(\text{GFA3CarryIStr}(x,y,z)) = \{\langle\langle x, y\rangle, \text{and}_{2b}\rangle, \langle\langle y,z\rangle, \text{and}_{2b}\rangle, \langle\langle z,x\rangle, \text{and}_{2b}\rangle\}$.

(124) For all sets x, y, z holds $\text{InnerVertices}(\text{GFA3CarryStr}(x,y,z)) = \{\langle\langle x, y\rangle, \text{and}_{2b}\rangle, \langle\langle y,z\rangle, \text{and}_{2b}\rangle, \langle\langle z,x\rangle, \text{and}_{2b}\rangle\} \cup \{\text{GFA3CarryOutput}(x,y,z)\}$.

(125) For all sets x, y, z holds $\text{InnerVertices}(\text{GFA3CarryStr}(x,y,z))$ is a binary relation.

(126) For all sets x, y, z such that $x \neq \langle\langle y,z\rangle, \text{and}_{2b}\rangle$ and $y \neq \langle\langle z,x\rangle, \text{and}_{2b}\rangle$ and $z \neq \langle\langle x,y\rangle, \text{and}_{2b}\rangle$ holds $\text{InputVertices}(\text{GFA3CarryIStr}(x,y,z)) = \{x,y,z\}$.

(127) For all sets x, y, z such that $x \neq \langle\langle y,z\rangle, \text{and}_{2b}\rangle$ and $y \neq \langle\langle z,x\rangle, \text{and}_{2b}\rangle$ and $z \neq \langle\langle x,y\rangle, \text{and}_{2b}\rangle$ holds $\text{InputVertices}(\text{GFA3CarryStr}(x,y,z)) = \{x,y,z\}$.

(128) For all non pair sets x, y, z holds $\text{InputVertices}(\text{GFA3CarryStr}(x,y,z))$ has no pairs.

(129) Let x, y, z be sets. Then $x \in$ the carrier of $\text{GFA3CarryStr}(x,y,z)$ and $y \in$ the carrier of $\text{GFA3CarryStr}(x,y,z)$ and $z \in$ the carrier of $\text{GFA3CarryStr}(x,y,z)$ and $\langle\langle x,y\rangle, \text{and}_{2b}\rangle \in$ the carrier of $\text{GFA3CarryStr}(x,y,z)$ and $\langle\langle y,z\rangle, \text{and}_{2b}\rangle \in$ the carrier of $\text{GFA3CarryStr}(x,y,z)$ and $\langle\langle z,x\rangle, \text{and}_{2b}\rangle \in$ the carrier of $\text{GFA3CarryStr}(x,y,z)$ and $\langle\langle\langle\langle x,y\rangle, \text{and}_{2b}\rangle, \langle\langle y,z\rangle, \text{and}_{2b}\rangle, \langle\langle z,x\rangle, \text{and}_{2b}\rangle\rangle, \text{nor}_3\rangle \in$ the carrier of $\text{GFA3CarryStr}(x,y,z)$.

(130) For all sets x, y, z holds $\langle\langle x,y\rangle, \text{and}_{2b}\rangle \in \text{InnerVertices}(\text{GFA3CarryStr}(x, y,z))$ and $\langle\langle y,z\rangle, \text{and}_{2b}\rangle \in \text{InnerVertices}(\text{GFA3CarryStr}(x,y,z))$ and $\langle\langle z, x\rangle, \text{and}_{2b}\rangle \in \text{InnerVertices}(\text{GFA3CarryStr}(x,y,z))$ and $\text{GFA3CarryOutput}(x,y,z) \in \text{InnerVertices}(\text{GFA3CarryStr}(x,y,z))$.

(131) For all sets x, y, z such that $x \neq \langle\langle y,z\rangle, \text{and}_{2b}\rangle$ and $y \neq \langle\langle z,x\rangle, \text{and}_{2b}\rangle$ and $z \neq \langle\langle x,y\rangle, \text{and}_{2b}\rangle$ holds $x \in \text{InputVertices}(\text{GFA3CarryStr}(x,y,z))$ and $y \in \text{InputVertices}(\text{GFA3CarryStr}(x,y,z))$ and

$z \in$ InputVertices(GFA3CarryStr(x, y, z)).

(132) For all non pair sets x, y, z holds InputVertices(GFA3CarryStr(x, y, z)) $= \{x, y, z\}$.

(133) Let x, y, z be sets, s be a state of GFA3CarryCirc(x, y, z), and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following(s))($\langle\!\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle\!\rangle$) $= \neg a_1 \wedge \neg a_2$ and (Following(s))($\langle\!\langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle\!\rangle$) $= \neg a_2 \wedge \neg a_3$ and (Following(s))($\langle\!\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\!\rangle$) $= \neg a_3 \wedge \neg a_1$.

(134) Let x, y, z be sets, s be a state of GFA3CarryCirc(x, y, z), and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(\langle\!\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle\!\rangle)$ and $a_2 = s(\langle\!\langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle\!\rangle)$ and $a_3 = s(\langle\!\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\!\rangle)$, then (Following$(s)$)(GFA3CarryOutput$(x, y, z)$) $= \neg(a_1 \vee a_2 \vee a_3)$.

(135) Let x, y, z be sets. Suppose $x \neq \langle\!\langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle\!\rangle$ and $y \neq \langle\!\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\!\rangle$ and $z \neq \langle\!\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle\!\rangle$. Let s be a state of GFA3CarryCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA3CarryOutput(x, y, z)) $= \neg(\neg a_1 \wedge \neg a_2 \vee \neg a_2 \wedge \neg a_3 \vee \neg a_3 \wedge \neg a_1)$ and (Following$(s, 2)$)($\langle\!\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle\!\rangle$) $= \neg a_1 \wedge \neg a_2$ and (Following$(s, 2)$)($\langle\!\langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle\!\rangle$) $= \neg a_2 \wedge \neg a_3$ and (Following$(s, 2)$)($\langle\!\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\!\rangle$) $= \neg a_3 \wedge \neg a_1$.

(136) For all sets x, y, z such that $x \neq \langle\!\langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle\!\rangle$ and $y \neq \langle\!\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\!\rangle$ and $z \neq \langle\!\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle\!\rangle$ and for every state s of GFA3CarryCirc(x, y, z) holds Following$(s, 2)$ is stable.

Let x, y, z be sets. The functor GFA3AdderStr(x, y, z) yields an unsplit non void strict non empty many sorted signature with arity held in gates and Boolean denotation held in gates and is defined by:

(Def. 46) GFA3AdderStr$(x, y, z) =$ 2GatesCircStr$(x, y, z, \mathrm{xor}_2)$.

Let x, y, z be sets. The functor GFA3AdderCirc(x, y, z) yielding a strict Boolean circuit of GFA3AdderStr(x, y, z) with denotation held in gates is defined by:

(Def. 47) GFA3AdderCirc$(x, y, z) =$ 2GatesCircuit$(x, y, z, \mathrm{xor}_2)$.

Let x, y, z be sets. The functor GFA3AdderOutput(x, y, z) yielding an element of InnerVertices(GFA3AdderStr(x, y, z)) is defined by:

(Def. 48) GFA3AdderOutput$(x, y, z) =$ 2GatesCircOutput$(x, y, z, \mathrm{xor}_2)$.

One can prove the following propositions:

(137) For all sets x, y, z holds InnerVertices(GFA3AdderStr(x, y, z)) $= \{\langle\!\langle\langle x, y\rangle, \mathrm{xor}_2\rangle\!\rangle\} \cup \{$GFA3AdderOutput$(x, y, z)\}$.

(138) For all sets x, y, z holds InnerVertices(GFA3AdderStr(x, y, z)) is a binary relation.

(139) For all sets x, y, z such that $z \neq \langle\!\langle\langle x, y\rangle, \mathrm{xor}_2\rangle\!\rangle$ holds InputVertices(GFA3AdderStr(x, y, z)) $= \{x, y, z\}$.

(140) For all non pair sets x, y, z holds InputVertices(GFA3AdderStr(x, y, z)) has no pairs.

(141) Let x, y, z be sets. Then

(i) $x \in$ the carrier of GFA3AdderStr(x, y, z),

(ii) $y \in$ the carrier of GFA3AdderStr(x, y, z),

(iii) $z \in$ the carrier of GFA3AdderStr(x, y, z),

(iv) $\langle\langle x, y\rangle, \text{xor}_2\rangle \in$ the carrier of GFA3AdderStr(x, y, z), and

(v) $\langle\langle\langle\langle x, y\rangle, \text{xor}_2\rangle, z\rangle, \text{xor}_2\rangle \in$ the carrier of GFA3AdderStr(x, y, z).

(142) For all sets x, y, z holds $\langle\langle x, y\rangle, \text{xor}_2\rangle \in$ InnerVertices(GFA3AdderStr(x, y, z)) and GFA3AdderOutput$(x, y, z) \in$ InnerVertices(GFA3AdderStr(x, y, z)).

(143) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \text{xor}_2\rangle$ holds $x \in$ InputVertices(GFA3AdderStr(x, y, z)) and
$y \in$ InputVertices(GFA3AdderStr(x, y, z)) and
$z \in$ InputVertices(GFA3AdderStr(x, y, z)).

(144) For all non pair sets x, y, z holds InputVertices(GFA3AdderStr(x, y, z)) $= \{x, y, z\}$.

(145) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \text{xor}_2\rangle$. Let s be a state of GFA3AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following(s))$(\langle\langle x, y\rangle, \text{xor}_2\rangle) = a_1 \oplus a_2$ and (Following(s))$(x) = a_1$ and (Following(s))$(y) = a_2$ and (Following(s))$(z) = a_3$.

(146) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \text{xor}_2\rangle$. Let s be a state of GFA3AdderCirc(x, y, z) and a_4, a_1, a_2, a_3 be elements of *Boolean*. If $a_4 = s(\langle\langle x, y\rangle, \text{xor}_2\rangle)$ and $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$, then (Following(s))(GFA3AdderOutput(x, y, z)) $= a_4 \oplus a_3$.

(147) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \text{xor}_2\rangle$. Let s be a state of GFA3AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then (Following$(s, 2)$)(GFA3AdderOutput(x, y, z)) $= a_1 \oplus a_2 \oplus a_3$ and (Following$(s, 2)$)$(\langle\langle x, y\rangle, \text{xor}_2\rangle) = a_1 \oplus a_2$ and (Following$(s, 2)$)$(x) = a_1$ and (Following$(s, 2)$)$(y) = a_2$ and (Following$(s, 2)$)$(z) = a_3$.

(148) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \text{xor}_2\rangle$. Let s be a state of GFA3AdderCirc(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. If $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$, then (Following$(s, 2)$)(GFA3AdderOutput(x, y, z)) $= \neg(\neg a_1 \oplus \neg a_2 \oplus \neg a_3)$.

(149) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \text{xor}_2\rangle$ and for every state s of GFA3AdderCirc(x, y, z) holds Following$(s, 2)$ is stable.

Let x, y, z be sets. The functor BitGFA3Str(x, y, z) yielding an unsplit non void strict non empty many sorted signature with arity held in gates and

Boolean denotation held in gates is defined by:

(Def. 49) $\text{BitGFA3Str}(x, y, z) = \text{GFA3AdderStr}(x, y, z) {+\cdot}\, \text{GFA3CarryStr}(x, y, z)$.

Let x, y, z be sets. The functor $\text{BitGFA3Circ}(x, y, z)$ yields a strict Boolean circuit of $\text{BitGFA3Str}(x, y, z)$ with denotation held in gates and is defined as follows:

(Def. 50) $\text{BitGFA3Circ}(x, y, z) = \text{GFA3AdderCirc}(x, y, z) {+\cdot}\, \text{GFA3CarryCirc}(x, y, z)$.

One can prove the following propositions:

(150) For all sets x, y, z holds $\text{InnerVertices}(\text{BitGFA3Str}(x, y, z)) = \{\langle\langle x, y\rangle, \text{xor}_2\rangle\} \cup \{\text{GFA3AdderOutput}(x, y, z)\} \cup \{\langle\langle x, y\rangle, \text{and}_{2b}\rangle, \langle\langle y, z\rangle, \text{and}_{2b}\rangle, \langle\langle z, x\rangle, \text{and}_{2b}\rangle\} \cup \{\text{GFA3CarryOutput}(x, y, z)\}$.

(151) For all sets x, y, z holds $\text{InnerVertices}(\text{BitGFA3Str}(x, y, z))$ is a binary relation.

(152) For all sets x, y, z such that $z \neq \langle\langle x, y\rangle, \text{xor}_2\rangle$ and $x \neq \langle\langle y, z\rangle, \text{and}_{2b}\rangle$ and $y \neq \langle\langle z, x\rangle, \text{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \text{and}_{2b}\rangle$ holds $\text{InputVertices}(\text{BitGFA3Str}(x, y, z)) = \{x, y, z\}$.

(153) For all non pair sets x, y, z holds $\text{InputVertices}(\text{BitGFA3Str}(x, y, z)) = \{x, y, z\}$.

(154) For all non pair sets x, y, z holds $\text{InputVertices}(\text{BitGFA3Str}(x, y, z))$ has no pairs.

(155) Let x, y, z be sets. Then $x \in$ the carrier of $\text{BitGFA3Str}(x, y, z)$ and $y \in$ the carrier of $\text{BitGFA3Str}(x, y, z)$ and $z \in$ the carrier of $\text{BitGFA3Str}(x, y, z)$ and $\langle\langle x, y\rangle, \text{xor}_2\rangle \in$ the carrier of $\text{BitGFA3Str}(x, y, z)$ and $\langle\langle\langle\langle x, y\rangle, \text{xor}_2\rangle, z\rangle, \text{xor}_2\rangle \in$ the carrier of $\text{BitGFA3Str}(x, y, z)$ and $\langle\langle x, y\rangle, \text{and}_{2b}\rangle \in$ the carrier of $\text{BitGFA3Str}(x, y, z)$ and $\langle\langle y, z\rangle, \text{and}_{2b}\rangle \in$ the carrier of $\text{BitGFA3Str}(x, y, z)$ and $\langle\langle z, x\rangle, \text{and}_{2b}\rangle \in$ the carrier of $\text{BitGFA3Str}(x, y, z)$ and $\langle\langle\langle\langle x, y\rangle, \text{and}_{2b}\rangle, \langle\langle y, z\rangle, \text{and}_{2b}\rangle, \langle\langle z, x\rangle, \text{and}_{2b}\rangle\rangle, \text{nor}_3\rangle \in$ the carrier of $\text{BitGFA3Str}(x, y, z)$.

(156) Let x, y, z be sets. Then $\langle\langle x, y\rangle, \text{xor}_2\rangle \in \text{InnerVertices}(\text{BitGFA3Str}(x, y, z))$ and $\text{GFA3AdderOutput}(x, y, z) \in \text{InnerVertices}(\text{BitGFA3Str}(x, y, z))$ and $\langle\langle x, y\rangle, \text{and}_{2b}\rangle \in \text{InnerVertices}(\text{BitGFA3Str}(x, y, z))$ and $\langle\langle y, z\rangle, \text{and}_{2b}\rangle \in \text{InnerVertices}(\text{BitGFA3Str}(x, y, z))$ and $\langle\langle z, x\rangle, \text{and}_{2b}\rangle \in \text{InnerVertices}(\text{BitGFA3Str}(x, y, z))$ and $\text{GFA3CarryOutput}(x, y, z) \in \text{InnerVertices}(\text{BitGFA3Str}(x, y, z))$.

(157) Let x, y, z be sets. Suppose $z \neq \langle\langle x, y\rangle, \text{xor}_2\rangle$ and $x \neq \langle\langle y, z\rangle, \text{and}_{2b}\rangle$ and $y \neq \langle\langle z, x\rangle, \text{and}_{2b}\rangle$ and $z \neq \langle\langle x, y\rangle, \text{and}_{2b}\rangle$. Then $x \in \text{InputVertices}(\text{BitGFA3Str}(x, y, z))$ and $y \in \text{InputVertices}(\text{BitGFA3Str}(x, y, z))$ and $z \in \text{InputVertices}(\text{BitGFA3Str}(x, y, z))$.

Let x, y, z be sets. The functor $\text{BitGFA3CarryOutput}(x, y, z)$ yields an element of $\text{InnerVertices}(\text{BitGFA3Str}(x, y, z))$ and is defined by:

(Def. 51) $\mathrm{BitGFA3CarryOutput}(x, y, z) = \langle\!\langle\langle\!\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle\!\rangle, \langle\!\langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle\!\rangle, \langle\!\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\!\rangle\rangle, \mathrm{nor}_3\rangle\!\rangle$.

Let x, y, z be sets. The functor BitGFA3AdderOutput(x, y, z) yielding an element of InnerVertices(BitGFA3Str(x, y, z)) is defined by:

(Def. 52) $\mathrm{BitGFA3AdderOutput}(x, y, z) = \mathrm{2GatesCircOutput}(x, y, z, \mathrm{xor}_2)$.

Next we state two propositions:

(158) Let x, y, z be sets. Suppose $z \neq \langle\!\langle\langle x, y\rangle, \mathrm{xor}_2\rangle\!\rangle$ and $x \neq \langle\!\langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle\!\rangle$ and $y \neq \langle\!\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\!\rangle$ and $z \neq \langle\!\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle\!\rangle$. Let s be a state of BitGFA3Circ(x, y, z) and a_1, a_2, a_3 be elements of *Boolean*. Suppose $a_1 = s(x)$ and $a_2 = s(y)$ and $a_3 = s(z)$. Then $(\mathrm{Following}(s, 2))(\mathrm{GFA3AdderOutput}(x, y, z)) = \neg(\neg a_1 \oplus \neg a_2 \oplus \neg a_3)$ and $(\mathrm{Following}(s, 2))(\mathrm{GFA3CarryOutput}(x, y, z)) = \neg(\neg a_1 \wedge \neg a_2 \vee \neg a_2 \wedge \neg a_3 \vee \neg a_3 \wedge \neg a_1)$.

(159) Let x, y, z be sets. Suppose $z \neq \langle\!\langle\langle x, y\rangle, \mathrm{xor}_2\rangle\!\rangle$ and $x \neq \langle\!\langle\langle y, z\rangle, \mathrm{and}_{2b}\rangle\!\rangle$ and $y \neq \langle\!\langle\langle z, x\rangle, \mathrm{and}_{2b}\rangle\!\rangle$ and $z \neq \langle\!\langle\langle x, y\rangle, \mathrm{and}_{2b}\rangle\!\rangle$. Let s be a state of BitGFA3Circ(x, y, z). Then Following$(s, 2)$ is stable.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[3] Grzegorz Bancerek and Yatsuka Nakamura. Full adder circuit. Part I. *Formalized Mathematics*, 5(**3**):367–380, 1996.
[4] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[9] Yatsuka Nakamura and Grzegorz Bancerek. Combining of circuits. *Formalized Mathematics*, 5(**2**):283–295, 1996.
[10] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Introduction to circuits, II. *Formalized Mathematics*, 5(**2**):273–278, 1996.
[11] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, II. *Formalized Mathematics*, 5(**2**):215–220, 1996.
[12] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.
[13] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[14] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[16] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[17] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.
[18] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[19] Katsumi Wasaki and Pauline N. Kawamoto. 2's complement circuit. *Formalized Mathematics*, 6(**2**):189–197, 1997.

[20] Edmund Woronowicz. Many–argument relations. *Formalized Mathematics*, 1(**4**):733–737, 1990.
[21] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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