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Inferior Limit and Superior Limit of Sequences of Real Numbers

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Summary. The concept of inferior limit and superior limit of sequences of real numbers is defined here. This article contains the following items: definition of the superior sequence and the inferior sequence of real numbers, definition of the superior limit and the inferior limit of real number, and definition of the relation between the limit value and the superior limit, the inferior limit of sequences of real numbers.

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The articles [2], [12], [6], [1], [3], [13], [10], [8], [15], [9], [16], [4], [14], [5], [11], and [7] provide the terminology and notation for this paper.

We adopt the following rules: n, m, k denote natural numbers, r, s, t denote real numbers, and s_1, s_2, s_3 denote sequences of real numbers.

One can prove the following proposition

(1) $s - r < t$ and $s + r > t$ iff $|t - s| < r$.

Let s_1 be a sequence of real numbers. The functor $\sup s_1$ yielding a real number is defined by:

(Def. 1) $\sup s_1 = \sup \operatorname{rng} s_1$.

Let s_1 be a sequence of real numbers. The functor $\inf s_1$ yielding a real number is defined as follows:

(Def. 2) $\inf s_1 = \inf \operatorname{rng} s_1$.

The following propositions are true:

(2) $(s_2 + s_3) - s_3 = s_2$.

(3) $r \in \operatorname{rng} s_1$ iff $-r \in \operatorname{rng}(-s_1)$.

(4) $\operatorname{rng}(-s_1) = -\operatorname{rng} s_1$.

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(5) s_1 is upper bounded iff $\operatorname{rng} s_1$ is upper bounded.

(6) s_1 is lower bounded iff $\operatorname{rng} s_1$ is lower bounded.

(7) Suppose s_1 is upper bounded. Then $r = \sup s_1$ if and only if the following conditions are satisfied:

(i) for every n holds $s_1(n) \leq r$, and

(ii) for every s such that $0 < s$ there exists k such that $r - s < s_1(k)$.

(8) Suppose s_1 is lower bounded. Then $r = \inf s_1$ if and only if the following conditions are satisfied:

(i) for every n holds $r \leq s_1(n)$, and

(ii) for every s such that $0 < s$ there exists k such that $s_1(k) < r + s$.

(9) For every n holds $s_1(n) \leq r$ iff s_1 is upper bounded and $\sup s_1 \leq r$.

(10) For every n holds $r \leq s_1(n)$ iff s_1 is lower bounded and $r \leq \inf s_1$.

(11) s_1 is upper bounded iff $-s_1$ is lower bounded.

(12) s_1 is lower bounded iff $-s_1$ is upper bounded.

(13) If s_1 is upper bounded, then $\sup s_1 = -\inf(-s_1)$.

(14) If s_1 is lower bounded, then $\inf s_1 = -\sup(-s_1)$.

(15) If s_2 is lower bounded and s_3 is lower bounded, then $\inf(s_2 + s_3) \geq \inf s_2 + \inf s_3$.

(16) If s_2 is upper bounded and s_3 is upper bounded, then $\sup(s_2 + s_3) \leq \sup s_2 + \sup s_3$.

Let f be a sequence of real numbers. We introduce f is non-negative as a synonym of f is non-negative yielding.

Let f be a sequence of real numbers. Let us observe that f is non-negative if and only if:

(Def. 3) For every n holds $f(n) \geq 0$.

The following propositions are true:

(17) If s_1 is non-negative, then $s_1 \uparrow k$ is non-negative.

(18) If s_1 is lower bounded and non-negative, then $\inf s_1 \geq 0$.

(19) If s_1 is upper bounded and non-negative, then $\sup s_1 \geq 0$.

(20) Suppose s_2 is lower bounded and non-negative and s_3 is lower bounded and non-negative. Then $s_2\, s_3$ is lower bounded and $\inf(s_2\, s_3) \geq \inf s_2 \cdot \inf s_3$.

(21) Suppose s_2 is upper bounded and non-negative and s_3 is upper bounded and non-negative. Then $s_2\, s_3$ is upper bounded and $\sup(s_2\, s_3) \leq \sup s_2 \cdot \sup s_3$.

(22) If s_1 is non-decreasing and upper bounded, then s_1 is bounded.

(23) If s_1 is non-increasing and lower bounded, then s_1 is bounded.

(24) If s_1 is non-decreasing and upper bounded, then $\lim s_1 = \sup s_1$.

(25) If s_1 is non-increasing and lower bounded, then $\lim s_1 = \inf s_1$.

(26) If s_1 is upper bounded, then $s_1 \uparrow k$ is upper bounded.

(27) If s_1 is lower bounded, then $s_1 \uparrow k$ is lower bounded.

(28) If s_1 is bounded, then $s_1 \uparrow k$ is bounded.

(29) For all s_1, n holds $\{s_1(k) : n \leq k\}$ is a subset of $\mathbb{R}$.

(30) $\mathrm{rng}(s_1 \uparrow k) = \{s_1(n) : k \leq n\}$.

(31) If s_1 is upper bounded, then for every n and for every subset R of $\mathbb{R}$ such that $R = \{s_1(k) : n \leq k\}$ holds R is upper bounded.

(32) If s_1 is lower bounded, then for every n and for every subset R of $\mathbb{R}$ such that $R = \{s_1(k) : n \leq k\}$ holds R is lower bounded.

(33) If s_1 is bounded, then for every n and for every subset R of $\mathbb{R}$ such that $R = \{s_1(k) : n \leq k\}$ holds R is bounded.

(34) If s_1 is non-decreasing, then for every n and for every subset R of $\mathbb{R}$ such that $R = \{s_1(k) : n \leq k\}$ holds $\inf R = s_1(n)$.

(35) If s_1 is non-increasing, then for every n and for every subset R of $\mathbb{R}$ such that $R = \{s_1(k) : n \leq k\}$ holds $\sup R = s_1(n)$.

(36) Let given s_1. Then there exists a function f from $\mathbb{N}$ into $\mathbb{R}$ such that for every n and for every subset Y of $\mathbb{R}$ if $Y = \{s_1(k) : n \leq k\}$, then $f(n) = \sup Y$.

(37) Let given s_1. Then there exists a function f from $\mathbb{N}$ into $\mathbb{R}$ such that for every n and for every subset Y of $\mathbb{R}$ if $Y = \{s_1(k) : n \leq k\}$, then $f(n) = \inf Y$.

Let s_1 be a sequence of real numbers. The inferior realsequence s_1 yields a sequence of real numbers and is defined as follows:

(Def. 4) For every n and for every subset Y of $\mathbb{R}$ such that $Y = \{s_1(k) : n \leq k\}$ holds (the inferior realsequence s_1)$(n) = \inf Y$.

Let s_1 be a sequence of real numbers. The superior realsequence s_1 yields a sequence of real numbers and is defined by:

(Def. 5) For every n and for every subset Y of $\mathbb{R}$ such that $Y = \{s_1(k) : n \leq k\}$ holds (the superior realsequence s_1)$(n) = \sup Y$.

Next we state a number of propositions:

(38) (The inferior realsequence s_1)$(n) = \inf(s_1 \uparrow n)$.

(39) (The superior realsequence s_1)$(n) = \sup(s_1 \uparrow n)$.

(40) If s_1 is lower bounded, then (the inferior realsequence s_1)$(0) = \inf s_1$.

(41) If s_1 is upper bounded, then (the superior realsequence s_1)$(0) = \sup s_1$.

(42) Suppose s_1 is lower bounded. Then $r =$ (the inferior realsequence s_1)(n) if and only if for every k holds $r \leq s_1(n + k)$ and for every s such that $0 < s$ there exists k such that $s_1(n + k) < r + s$.

(43) Suppose s_1 is upper bounded. Then $r =$ (the superior realsequence $s_1)(n)$ if and only if for every k holds $s_1(n+k) \leq r$ and for every s such that $0 < s$ there exists k such that $r - s < s_1(n+k)$.

(44) If s_1 is lower bounded, then for every k holds $r \leq s_1(n+k)$ iff $r \leq$ (the inferior realsequence $s_1)(n)$.

(45) Suppose s_1 is lower bounded. Then for every m such that $n \leq m$ holds $r \leq s_1(m)$ if and only if $r \leq$ (the inferior realsequence $s_1)(n)$.

(46) If s_1 is upper bounded, then for every k holds $s_1(n+k) \leq r$ iff (the superior realsequence $s_1)(n) \leq r$.

(47) Suppose s_1 is upper bounded. Then for every m such that $n \leq m$ holds $s_1(m) \leq r$ if and only if (the superior realsequence $s_1)(n) \leq r$.

(48) If s_1 is lower bounded, then (the inferior realsequence $s_1)(n) = \min(($the inferior realsequence $s_1)(n+1), s_1(n))$.

(49) If s_1 is upper bounded, then (the superior realsequence $s_1)(n) = \max(($the superior realsequence $s_1)(n+1), s_1(n))$.

(50) If s_1 is lower bounded, then (the inferior realsequence $s_1)(n) \leq$ (the inferior realsequence $s_1)(n+1)$.

(51) If s_1 is upper bounded, then (the superior realsequence $s_1)(n+1) \leq$ (the superior realsequence $s_1)(n)$.

(52) If s_1 is lower bounded, then the inferior realsequence s_1 is non-decreasing.

(53) If s_1 is upper bounded, then the superior realsequence s_1 is non-increasing.

(54) If s_1 is bounded, then (the inferior realsequence $s_1)(n) \leq$ (the superior realsequence $s_1)(n)$.

(55) If s_1 is bounded, then (the inferior realsequence $s_1)(n) \leq \inf$ (the superior realsequence s_1).

(56) If s_1 is bounded, then $\sup$ (the inferior realsequence $s_1) \leq$ (the superior realsequence $s_1)(n)$.

(57) If s_1 is bounded, then $\sup$ (the inferior realsequence $s_1) \leq \inf$ (the superior realsequence s_1).

(58) If s_1 is bounded, then the superior realsequence s_1 is bounded and the inferior realsequence s_1 is bounded.

(59) Suppose s_1 is bounded. Then
  (i) the inferior realsequence s_1 is convergent, and
  (ii) $\lim$ (the inferior realsequence $s_1) = \sup$ (the inferior realsequence s_1).

(60) Suppose s_1 is bounded. Then
  (i) the superior realsequence s_1 is convergent, and
  (ii) $\lim$ (the superior realsequence $s_1) = \inf$ (the superior realsequence s_1).

(61) If s_1 is lower bounded, then (the inferior realsequence $s_1)(n)$ $=$ $-$(the superior realsequence $-s_1)(n)$.

(62) If s_1 is upper bounded, then (the superior realsequence $s_1)(n)$ $=$ $-$(the inferior realsequence $-s_1)(n)$.

(63) If s_1 is lower bounded, then the inferior realsequence s_1 $=$ $-$the superior realsequence $-s_1$.

(64) If s_1 is upper bounded, then the superior realsequence s_1 $=$ $-$the inferior realsequence $-s_1$.

(65) If s_1 is non-decreasing, then $s_1(n) \leq$ (the inferior realsequence $s_1)(n+1)$.

(66) If s_1 is non-decreasing, then the inferior realsequence $s_1 = s_1$.

(67) If s_1 is non-decreasing and upper bounded, then $s_1(n) \leq$ (the superior realsequence $s_1)(n + 1)$.

(68) Suppose s_1 is non-decreasing and upper bounded. Then (the superior realsequence $s_1)(n) =$ (the superior realsequence $s_1)(n + 1)$.

(69) Suppose s_1 is non-decreasing and upper bounded. Then (the superior realsequence $s_1)(n) = \sup s_1$ and the superior realsequence s_1 is constant.

(70) If s_1 is non-decreasing and upper bounded, then inf (the superior realsequence $s_1) = \sup s_1$.

(71) If s_1 is non-increasing, then (the superior realsequence $s_1)(n+1) \leq s_1(n)$.

(72) If s_1 is non-increasing, then the superior realsequence $s_1 = s_1$.

(73) If s_1 is non-increasing and lower bounded, then (the inferior realsequence $s_1)(n + 1) \leq s_1(n)$.

(74) Suppose s_1 is non-increasing and lower bounded. Then (the inferior realsequence $s_1)(n) =$ (the inferior realsequence $s_1)(n + 1)$.

(75) Suppose s_1 is non-increasing and lower bounded. Then (the inferior realsequence $s_1)(n) = \inf s_1$ and the inferior realsequence s_1 is constant.

(76) If s_1 is non-increasing and lower bounded, then sup (the inferior realsequence $s_1) = \inf s_1$.

(77) Suppose s_2 is bounded and s_3 is bounded and for every n holds $s_2(n) \leq s_3(n)$. Then

(i) for every n holds (the superior realsequence $s_2)(n) \leq$ (the superior realsequence $s_3)(n)$, and

(ii) for every n holds (the inferior realsequence $s_2)(n) \leq$ (the inferior realsequence $s_3)(n)$.

(78) Suppose s_2 is lower bounded and s_3 is lower bounded. Then (the inferior realsequence $s_2 + s_3)(n) \geq$ (the inferior realsequence $s_2)(n) +$ (the inferior realsequence $s_3)(n)$.

(79) Suppose s_2 is upper bounded and s_3 is upper bounded. Then (the superior realsequence $s_2 + s_3)(n) \leq$ (the superior realsequence $s_2)(n) +$ (the

superior realsequence $s_3)(n)$.

(80) Suppose s_2 is lower bounded and non-negative and s_3 is lower bounded and non-negative. Then (the inferior realsequence $s_2\, s_3)(n) \geq$ (the inferior realsequence $s_2)(n) \cdot$ (the inferior realsequence $s_3)(n)$.

(81) Suppose s_2 is lower bounded and non-negative and s_3 is lower bounded and non-negative. Then (the inferior realsequence $s_2\, s_3)(n) \geq$ (the inferior realsequence $s_2)(n) \cdot$ (the inferior realsequence $s_3)(n)$.

(82) Suppose s_2 is upper bounded and non-negative and s_3 is upper bounded and non-negative. Then (the superior realsequence $s_2\, s_3)(n) \leq$ (the superior realsequence $s_2)(n) \cdot$ (the superior realsequence $s_3)(n)$.

Let s_1 be a sequence of real numbers. The functor $\limsup s_1$ yielding an element of $\mathbb{R}$ is defined as follows:

(Def. 6) $\limsup s_1 = \inf$ (the superior realsequence s_1).

Let s_1 be a sequence of real numbers. The functor $\liminf s_1$ yielding an element of $\mathbb{R}$ is defined by:

(Def. 7) $\liminf s_1 = \sup$ (the inferior realsequence s_1).

Next we state a number of propositions:

(83) If s_1 is bounded, then $\liminf s_1 \leq r$ iff for every s such that $0 < s$ and for every n there exists k such that $s_1(n + k) < r + s$.

(84) If s_1 is bounded, then $r \leq \liminf s_1$ iff for every s such that $0 < s$ there exists n such that for every k holds $r - s < s_1(n + k)$.

(85) Suppose s_1 is bounded. Then $r = \liminf s_1$ if and only if for every s such that $0 < s$ holds for every n there exists k such that $s_1(n+k) < r+s$ and there exists n such that for every k holds $r - s < s_1(n + k)$.

(86) If s_1 is bounded, then $r \leq \limsup s_1$ iff for every s such that $0 < s$ and for every n there exists k such that $s_1(n + k) > r - s$.

(87) If s_1 is bounded, then $\limsup s_1 \leq r$ iff for every s such that $0 < s$ there exists n such that for every k holds $s_1(n + k) < r + s$.

(88) Suppose s_1 is bounded. Then $r = \limsup s_1$ if and only if for every s such that $0 < s$ holds for every n there exists k such that $s_1(n+k) > r-s$ and there exists n such that for every k holds $s_1(n + k) < r + s$.

(89) If s_1 is bounded, then $\liminf s_1 \leq \limsup s_1$.

(90) s_1 is bounded and $\limsup s_1 = \liminf s_1$ iff s_1 is convergent.

(91) If s_1 is convergent, then $\lim s_1 = \limsup s_1$ and $\lim s_1 = \liminf s_1$.

(92) If s_1 is bounded, then $\limsup(-s_1) = -\liminf s_1$ and $\liminf(-s_1) = -\limsup s_1$.

(93) If s_2 is bounded and s_3 is bounded and for every n holds $s_2(n) \leq s_3(n)$, then $\limsup s_2 \leq \limsup s_3$ and $\liminf s_2 \leq \liminf s_3$.

(94) Suppose s_2 is bounded and s_3 is bounded. Then $\liminf s_2 + \liminf s_3 \leq \liminf(s_2+s_3)$ and $\liminf(s_2+s_3) \leq \liminf s_2+\limsup s_3$ and $\liminf(s_2+s_3) \leq \limsup s_2+\liminf s_3$ and $\liminf s_2+\limsup s_3 \leq \limsup(s_2+s_3)$ and $\limsup s_2+\liminf s_3 \leq \limsup(s_2+s_3)$ and $\limsup(s_2+s_3) \leq \limsup s_2+\limsup s_3$ and if s_2 is convergent or s_3 is convergent, then $\liminf(s_2+s_3) = \liminf s_2 + \liminf s_3$ and $\limsup(s_2 + s_3) = \limsup s_2 + \limsup s_3$.

(95) If s_2 is bounded and non-negative and s_3 is bounded and non-negative, then $\liminf s_2 \cdot \liminf s_3 \leq \liminf(s_2\, s_3)$ and $\limsup(s_2\, s_3) \leq \limsup s_2 \cdot \limsup s_3$.

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