FORMALIZED MATHEMATICS
Volume 13, Number 3, Pages 367–373
University of Białystok, 2005

The Inner Product and Conjugate of Finite Sequences of Complex Numbers

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Summary. The concept of "the inner product and conjugate of finite sequences of complex numbers" is defined here. Addition, subtraction, scalar multiplication and inner product are introduced using correspondent definitions of "conjugate of finite sequences of field". Many equations for such operations consist like a case of "conjugate of finite sequences of field". Some operations on the set of n-tuples of complex numbers are introduced as well. Additionally, difference of such n-tuples, complement of a n-tuple and multiplication of these are defined in terms of complex numbers.

MML identifier: `COMPLSP2`, version: `7.5.01 4.39.921`

The terminology and notation used here are introduced in the following articles: [17], [18], [15], [19], [8], [9], [10], [4], [16], [3], [5], [12], [6], [11], [7], [14], [1], [2], and [13].

1. Preliminaries

For simplicity, we adopt the following convention: i, j are natural numbers, x, y, z are finite sequences of elements of $\mathbb{C}$, c is an element of $\mathbb{C}$, and R, R_1, R_2 are elements of $\mathbb{C}^i$.

Let z be a finite sequence of elements of $\mathbb{C}$. The functor $\overline{z}$ yielding a finite sequence of elements of $\mathbb{C}$ is defined by:

(Def. 1) $\operatorname{len}\overline{z} = \operatorname{len} z$ and for every natural number i such that $1 \leq i$ and $i \leq \operatorname{len} z$ holds $\overline{z}(i) = \overline{z(i)}$.

The following propositions are true:

ISSN 1426–2630

(1) If $i \in \operatorname{dom}(x+y)$, then $(x+y)(i) = x(i) + y(i)$.

(2) If $i \in \operatorname{dom}(x-y)$, then $(x-y)(i) = x(i) - y(i)$.

Let us consider i, R_1, R_2. Then $R_1 - R_2$ is an element of $\mathbb{C}^i$.

Let us consider i, R_1, R_2. Then $R_1 + R_2$ is an element of $\mathbb{C}^i$.

Let us consider i, let r be a complex number, and let us consider R. Then $r \cdot R$ is an element of $\mathbb{C}^i$.

We now state a number of propositions:

(3) For every complex number a and for every finite sequence x of elements of $\mathbb{C}$ holds $\operatorname{len}(a \cdot x) = \operatorname{len} x$.

(4) For every finite sequence x of elements of $\mathbb{C}$ holds $\operatorname{dom} x = \operatorname{dom}(-x)$.

(5) For every finite sequence x of elements of $\mathbb{C}$ holds $\operatorname{len}(-x) = \operatorname{len} x$.

(6) For all finite sequences x_1, x_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $\operatorname{len}(x_1 + x_2) = \operatorname{len} x_1$.

(7) For all finite sequences x_1, x_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $\operatorname{len}(x_1 - x_2) = \operatorname{len} x_1$.

(8) Every finite sequence f of elements of $\mathbb{C}$ is an element of $\mathbb{C}^{\operatorname{len} f}$.

(9) $R_1 - R_2 = R_1 + -R_2$.

(10) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $x - y = x + -y$.

(11) $(-1) \cdot R = -R$.

(12) For every finite sequence x of elements of $\mathbb{C}$ holds $(-1) \cdot x = -x$.

(13) For every finite sequence x of elements of $\mathbb{C}$ holds $(-x)(i) = -x(i)$.

Let us consider i, R. Then $-R$ is an element of $\mathbb{C}^i$.

The following propositions are true:

(14) If $c = R(j)$, then $(-R)(j) = -c$.

(15) For every complex number a holds $\operatorname{dom}(a \cdot x) = \operatorname{dom} x$.

(16) For every complex number a holds $(a \cdot x)(i) = a \cdot x(i)$.

(17) For every complex number a holds $\overline{a \cdot x} = \overline{a} \cdot \overline{x}$.

(18) $(R_1 + R_2)(j) = R_1(j) + R_2(j)$.

(19) For all finite sequences x_1, x_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $\overline{x_1 + x_2} = \overline{x_1} + \overline{x_2}$.

(20) $(R_1 - R_2)(j) = R_1(j) - R_2(j)$.

(21) For all finite sequences x_1, x_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $\overline{x_1 - x_2} = \overline{x_1} - \overline{x_2}$.

(22) For every finite sequence z of elements of $\mathbb{C}$ holds $\overline{\overline{z}} = z$.

(23) For every finite sequence z of elements of $\mathbb{C}$ holds $\overline{-z} = -\overline{z}$.

(24) For every complex number z holds $z + \overline{z} = 2 \cdot \Re(z)$.

(25) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $(x - y)(i) = x(i) - y(i)$.

(26) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $(x + y)(i) = x(i) + y(i)$.

Let z be a finite sequence of elements of $\mathbb{C}$. The functor $\Re(z)$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def. 2) $\Re(z) = \frac{1}{2} \cdot (z + \overline{z})$.

One can prove the following proposition

(27) For every complex number z holds $z - \overline{z} = 2 \cdot \Im(z) \cdot i$.

Let z be a finite sequence of elements of $\mathbb{C}$. The functor $\Im(z)$ yielding a finite sequence of elements of $\mathbb{R}$ is defined as follows:

(Def. 3) $\Im(z) = (-\frac{1}{2} \cdot i) \cdot (z - \overline{z})$.

Let x, y be finite sequences of elements of $\mathbb{C}$. The functor $|(x, y)|$ yields an element of $\mathbb{C}$ and is defined by:

(Def. 4) $|(x, y)| = (|(\Re(x), \Re(y))| - i \cdot |(\Re(x), \Im(y))|) + i \cdot |(\Im(x), \Re(y))| + |(\Im(x), \Im(y))|$.

We now state four propositions:

(28) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $x + (y + z) = (x + y) + z$.

(29) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $x + y = y + x$.

(30) Let c be a complex number and x, y be finite sequences of elements of $\mathbb{C}$. If $\operatorname{len} x = \operatorname{len} y$, then $c \cdot (x + y) = c \cdot x + c \cdot y$.

(31) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $x - y = x + -y$.

Let us consider i, c. Then $i \mapsto c$ is an element of $\mathbb{C}^i$.

Next we state a number of propositions:

(32) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $x + y = 0_{\mathbb{C}}^{\operatorname{len} x}$ holds $x = -y$ and $y = -x$.

(33) For every finite sequence x of elements of $\mathbb{C}$ holds $x + 0_{\mathbb{C}}^{\operatorname{len} x} = x$.

(34) For every finite sequence x of elements of $\mathbb{C}$ holds $x + -x = 0_{\mathbb{C}}^{\operatorname{len} x}$.

(35) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $-(x + y) = -x + -y$.

(36) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $x - y - z = x - (y + z)$.

(37) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $x + (y - z) = (x + y) - z$.

(38) For every finite sequence x of elements of $\mathbb{C}$ holds $--x = x$.

(39) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $-(x - y) = -x + y$.

(40) For all finite sequences x, y, z of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$ holds $x - (y - z) = (x - y) + z$.

(41) For every complex number c holds $c \cdot 0_{\mathbb{C}}^{\operatorname{len} x} = 0_{\mathbb{C}}^{\operatorname{len} x}$.

(42) For every complex number c holds $-c \cdot x = c \cdot -x$.

(43) Let c be a complex number and x, y be finite sequences of elements of $\mathbb{C}$. If $\operatorname{len} x = \operatorname{len} y$, then $c \cdot (x - y) = c \cdot x - c \cdot y$.

(44) For all elements x_1, y_1 of $\mathbb{C}$ and for all real numbers x_2, y_2 such that $x_1 = x_2$ and $y_1 = y_2$ holds $+_{\mathbb{C}}(x_1, y_1) = +_{\mathbb{R}}(x_2, y_2)$.

In the sequel C is a function from $[\colon \mathbb{C}, \mathbb{C} \colon]$ into $\mathbb{C}$ and G is a function from $[\colon \mathbb{R}, \mathbb{R} \colon]$ into $\mathbb{R}$.

One can prove the following proposition

(45) Let x_1, y_1 be finite sequences of elements of $\mathbb{C}$ and x_2, y_2 be finite sequences of elements of $\mathbb{R}$. Suppose $x_1 = x_2$ and $y_1 = y_2$ and $\operatorname{len} x_1 = \operatorname{len} y_2$ and for every i such that $i \in \operatorname{dom} x_1$ holds $C(x_1(i), y_1(i)) = G(x_2(i), y_2(i))$. Then $C^\circ(x_1, y_1) = G^\circ(x_2, y_2)$.

Let z be a finite sequence of elements of $\mathbb{R}$ and let i be a set. Then $z(i)$ is an element of $\mathbb{R}$.

We now state several propositions:

(46) Let x_1, y_1 be finite sequences of elements of $\mathbb{C}$ and x_2, y_2 be finite sequences of elements of $\mathbb{R}$. If $x_1 = x_2$ and $y_1 = y_2$ and $\operatorname{len} x_1 = \operatorname{len} y_2$, then $(+_{\mathbb{C}})^\circ(x_1, y_1) = (+_{\mathbb{R}})^\circ(x_2, y_2)$.

(47) Let x_1, y_1 be finite sequences of elements of $\mathbb{C}$ and x_2, y_2 be finite sequences of elements of $\mathbb{R}$. If $x_1 = x_2$ and $y_1 = y_2$ and $\operatorname{len} x_1 = \operatorname{len} y_2$, then $x_1 + y_1 = x_2 + y_2$.

(48) For every finite sequence x of elements of $\mathbb{C}$ holds $\operatorname{len} \Re(x) = \operatorname{len} x$ and $\operatorname{len} \Im(x) = \operatorname{len} x$.

(49) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $\Re(x + y) = \Re(x) + \Re(y)$ and $\Im(x + y) = \Im(x) + \Im(y)$.

(50) Let x_1, y_1 be finite sequences of elements of $\mathbb{C}$ and x_2, y_2 be finite sequences of elements of $\mathbb{R}$. If $x_1 = x_2$ and $y_1 = y_2$ and $\operatorname{len} x_1 = \operatorname{len} y_2$, then $(-_{\mathbb{C}})^\circ(x_1, y_1) = (-_{\mathbb{R}})^\circ(x_2, y_2)$.

(51) Let x_1, y_1 be finite sequences of elements of $\mathbb{C}$ and x_2, y_2 be finite sequences of elements of $\mathbb{R}$. If $x_1 = x_2$ and $y_1 = y_2$ and $\operatorname{len} x_1 = \operatorname{len} y_2$, then $x_1 - y_1 = x_2 - y_2$.

(52) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $\Re(x - y) = \Re(x) - \Re(y)$ and $\Im(x - y) = \Im(x) - \Im(y)$.

(53) For all complex numbers a, b holds $a \cdot (b \cdot z) = (a \cdot b) \cdot z$.

(54) For every complex number c holds $(-c) \cdot x = -c \cdot x$.

In the sequel h is a function from $\mathbb{C}$ into $\mathbb{C}$ and g is a function from $\mathbb{R}$ into $\mathbb{R}$.

One can prove the following propositions:

(55) Let y_1 be a finite sequence of elements of $\mathbb{C}$ and y_2 be a finite sequence of elements of $\mathbb{R}$. If $\operatorname{len} y_1 = \operatorname{len} y_2$ and for every i such that $i \in \operatorname{dom} y_1$ holds $h(y_1(i)) = g(y_2(i))$, then $h \cdot y_1 = g \cdot y_2$.

(56) Let y_1 be a finite sequence of elements of $\mathbb{C}$ and y_2 be a finite sequence of elements of $\mathbb{R}$. If $y_1 = y_2$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $-_{\mathbb{C}} \cdot y_1 = -_{\mathbb{R}} \cdot y_2$.

(57) Let y_1 be a finite sequence of elements of $\mathbb{C}$ and y_2 be a finite sequence of elements of $\mathbb{R}$. If $y_1 = y_2$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $-y_1 = -y_2$.

(58) For every finite sequence x of elements of $\mathbb{C}$ holds $\Re(i \cdot x) = -\Im(x)$ and $\Im(i \cdot x) = \Re(x)$.

(59) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(i \cdot x, y)| = i \cdot |(x, y)|$.

(60) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x, i \cdot y)| = -i \cdot |(x, y)|$.

(61) Let a_1 be an element of $\mathbb{C}$, y_1 be a finite sequence of elements of $\mathbb{C}$, a_2 be an element of $\mathbb{R}$, and y_2 be a finite sequence of elements of $\mathbb{R}$. If $a_1 = a_2$ and $y_1 = y_2$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $\cdot_{\mathbb{C}}^{(a_1)} \cdot y_1 = \cdot_{\mathbb{R}}^{a_2} \cdot y_2$.

(62) Let a_1 be a complex number, y_1 be a finite sequence of elements of $\mathbb{C}$, a_2 be an element of $\mathbb{R}$, and y_2 be a finite sequence of elements of $\mathbb{R}$. If $a_1 = a_2$ and $y_1 = y_2$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $a_1 \cdot y_1 = a_2 \cdot y_2$.

(63) For all complex numbers a, b holds $(a + b) \cdot z = a \cdot z + b \cdot z$.

(64) For all complex numbers a, b holds $(a - b) \cdot z = a \cdot z - b \cdot z$.

(65) Let a be an element of $\mathbb{C}$ and x be a finite sequence of elements of $\mathbb{C}$. Then $\Re(a \cdot x) = \Re(a) \cdot \Re(x) - \Im(a) \cdot \Im(x)$ and $\Im(a \cdot x) = \Im(a) \cdot \Re(x) + \Re(a) \cdot \Im(x)$.

2. The Inner Product and Conjugate of Finite Sequences

The following propositions are true:

(66) For all finite sequences x_1, x_2, y of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} y$ holds $|(x_1 + x_2, y)| = |(x_1, y)| + |(x_2, y)|$.

(67) For all finite sequences x_1, x_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $|(-x_1, x_2)| = -|(x_1, x_2)|$.

(68) For all finite sequences x_1, x_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $|(x_1, -x_2)| = -|(x_1, x_2)|$.

(69) For all finite sequences x_1, x_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ holds $|(-x_1, -x_2)| = |(x_1, x_2)|$.

(70) For all finite sequences x_1, x_2, x_3 of elements of $\mathbb{C}$ such that $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} x_3$ holds $|(x_1 - x_2, x_3)| = |(x_1, x_3)| - |(x_2, x_3)|$.

(71) For all finite sequences x, y_1, y_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$ holds $|(x, y_1 + y_2)| = |(x, y_1)| + |(x, y_2)|$.

(72) For all finite sequences x, y_1, y_2 of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$ holds $|(x, y_1 - y_2)| = |(x, y_1)| - |(x, y_2)|$.

(73) Let x_1, x_2, y_1, y_2 be finite sequences of elements of $\mathbb{C}$. If $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $|(x_1+x_2, y_1+y_2)| = |(x_1, y_1)| + |(x_1, y_2)| + |(x_2, y_1)| + |(x_2, y_2)|$.

(74) Let x_1, x_2, y_1, y_2 be finite sequences of elements of $\mathbb{C}$. If $\operatorname{len} x_1 = \operatorname{len} x_2$ and $\operatorname{len} x_2 = \operatorname{len} y_1$ and $\operatorname{len} y_1 = \operatorname{len} y_2$, then $|(x_1 - x_2, y_1 - y_2)| = (|(x_1, y_1)| - |(x_1, y_2)| - |(x_2, y_1)|) + |(x_2, y_2)|$.

(75) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x, y)| = \overline{|(y, x)|}$.

(76) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x + y, x + y)| = |(x, x)| + 2 \cdot \Re(|(x, y)|) + |(y, y)|$.

(77) For all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x - y, x - y)| = (|(x, x)| - 2 \cdot \Re(|(x, y)|)) + |(y, y)|$.

(78) For every element a of $\mathbb{C}$ and for all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(a \cdot x, y)| = a \cdot |(x, y)|$.

(79) For every element a of $\mathbb{C}$ and for all finite sequences x, y of elements of $\mathbb{C}$ such that $\operatorname{len} x = \operatorname{len} y$ holds $|(x, a \cdot y)| = \overline{a} \cdot |(x, y)|$.

(80) Let a, b be elements of $\mathbb{C}$ and x, y, z be finite sequences of elements of $\mathbb{C}$. If $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$, then $|(a \cdot x + b \cdot y, z)| = a \cdot |(x, z)| + b \cdot |(y, z)|$.

(81) Let a, b be elements of $\mathbb{C}$ and x, y, z be finite sequences of elements of $\mathbb{C}$. If $\operatorname{len} x = \operatorname{len} y$ and $\operatorname{len} y = \operatorname{len} z$, then $|(x, a \cdot y + b \cdot z)| = \overline{a} \cdot |(x, y)| + \overline{b} \cdot |(x, z)|$.

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Received April 25, 2005