Stirling Numbers of the Second Kind

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Summary. In this paper we define Stirling numbers of the second kind by cardinality of certain functional classes so that

$$S(n,k) = \{f \text{ where } f \text{ is function of } n, k : f \text{ is onto increasing}\}$$

After that we show basic properties of this number in order to prove recursive dependence of Stirling number of the second kind. Consecutive theorems are introduced to prove formula

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} (k-i)^n$$

where $k \leq n$.

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The papers [18], [9], [21], [14], [23], [6], [24], [2], [3], [8], [10], [1], [22], [7], [11], [20], [16], [19], [4], [5], [13], [12], [17], and [15] provide the terminology and notation for this paper.

For simplicity, we adopt the following convention: k, l, m, n, i, j denote natural numbers, K, N denote non empty subsets of $\mathbb{N}$, K_1, N_1, M_1 denote subsets of $\mathbb{N}$, and X, Y denote sets.

Let us consider k. Then $\{k\}$ is a subset of $\mathbb{N}$. Let us consider l. Then $\{k,l\}$ is a subset of $\mathbb{N}$. Let us consider m. Then $\{k,l,m\}$ is a non empty subset of $\mathbb{N}$.

The following propositions are true:

(1) $\min N = \min^* N$.

(2) $\min(\min K, \min N) = \min(K \cup N)$.

(3) $\min(\min^* K_1, \min^* N_1) \leq \min^*(K_1 \cup N_1)$.

(4)   If $\min^* N_1 \notin N_1 \cap K_1$, then $\min^* N_1 = \min^* (N_1 \setminus K_1)$.

(5) $\min^* \{n\} = n$ and $\min\{n\} = n$.

(6) $\min^* \{n, k\} = \min(n, k)$ and $\min\{n, k\} = \min(n, k)$.

(7) $\min^* \{n, k, l\} = \min(n, \min(k, l))$.

(8) n is a subset of $\mathbb{N}$.

Let us consider n. One can verify that every element of n is natural.

We now state several propositions:

(9)   If $N \subseteq n$, then $n - 1$ is a natural number.

(10)   If $k \in n$, then $k \leq n - 1$ and $n - 1$ is a natural number.

(11) $\min^* n = 0$.

(12)   If $N \subseteq n$, then $\min^* N \leq n - 1$.

(13)   If $N \subseteq n$ and $N \neq \{n - 1\}$, then $\min^* N < n - 1$.

(14)   If $N_1 \subseteq n$ and $n > 0$, then $\min^* N_1 \leq n - 1$.

In the sequel f, g are functions from n into k.

Let us consider n, X, let f be a function from n into X, and let x be a set. Then $f^{-1}(x)$ is a subset of $\mathbb{N}$.

Let us consider X, k, let f be a function from X into k, and let x be a set. Then $f(x)$ is an element of k.

Let us consider X, N_1, let f be a function from X into N_1, and let x be a set. One can verify that $f(x)$ is natural.

Let us consider n, k and let f be a function from n into k. We say that f is increasing if and only if:

(Def. 1) $n = 0$ iff $k = 0$ and for all l, m such that $l \in \operatorname{rng} f$ and $m \in \operatorname{rng} f$ and $l < m$ holds $\min^* (f^{-1}(\{l\})) < \min^* (f^{-1}(\{m\}))$.

We now state several propositions:

(15)   If $n = 0$ and $k = 0$, then f is onto and increasing.

(16)   If $n > 0$, then $\min^* (f^{-1}(\{m\})) \leq n - 1$.

(17)   If f is onto, then $n \geq k$.

(18)   If f is onto and increasing, then for every m such that $m < k$ holds $m \leq \min^* (f^{-1}(\{m\}))$.

(19)   If f is onto and increasing, then for every m such that $m < k$ holds $\min^* (f^{-1}(\{m\})) \leq (n - k) + m$.

(20)   If f is onto and increasing and $n = k$, then $f = \operatorname{id}_n$.

(21)   If $f = \operatorname{id}_n$ and $n > 0$, then f is increasing.

(22)   If $n = 0$ iff $k = 0$, then there exists a function from n into k which is increasing.

(23)   If $n = 0$ iff $k = 0$ and $n \geq k$, then there exists a function from n into k which is onto and increasing.

The scheme *Sch1* deals with natural numbers $\mathcal{A}$, $\mathcal{B}$ and a unary predicate $\mathcal{P}$, and states that:

$\{f; f$ ranges over functions from $\mathcal{A}$ into $\mathcal{B} : \mathcal{P}[f]\}$ is finite

for all values of the parameters.

In the sequel f is a function from n into k.

One can prove the following propositions:

(24) For all n, k holds $\{f : f$ is onto and increasing$\}$ is finite.

(25) For all n, k holds $\overline{\overline{\{f : f \text{ is onto and increasing}\}}}$ is a natural number.

Let us consider n, k. The functor $n\,\mathrm{block}\,k$ yields a natural number and is defined by:

(Def. 2) $\quad n\,\mathrm{block}\,k = \overline{\overline{\{f : f \text{ is onto and increasing}\}}}$.

Next we state several propositions:

(26) $n\,\mathrm{block}\,n = \mathbf{1}$.

(27) If $k \neq 0$, then $0\,\mathrm{block}\,k = 0$.

(28) $0\,\mathrm{block}\,k = \mathbf{1}$ iff $k = 0$.

(29) If $n < k$, then $n\,\mathrm{block}\,k = 0$.

(30) $n\,\mathrm{block}\,0 = \mathbf{1}$ iff $n = 0$.

(31) If $n \neq 0$, then $n\,\mathrm{block}\,0 = 0$.

(32) If $n \neq 0$, then $n\,\mathrm{block}\,1 = \mathbf{1}$.

(33) $1 \leq k$ and $k \leq n$ or $k = n$ iff $n\,\mathrm{block}\,k > 0$.

In the sequel x, y denote sets.

Now we present three schemes. The scheme *Sch2* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, a function $\mathcal{E}$ from $\mathcal{A}$ into $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a function h from $\mathcal{C}$ into $\mathcal{D}$ such that $h{\upharpoonright}\mathcal{A} = \mathcal{E}$ and

for every x such that $x \in \mathcal{C} \setminus \mathcal{A}$ holds $h(x) = \mathcal{F}(x)$

provided the parameters satisfy the following conditions:

- For every x such that $x \in \mathcal{C} \setminus \mathcal{A}$ holds $\mathcal{F}(x) \in \mathcal{D}$,
- $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{B} \subseteq \mathcal{D}$, and
- If $\mathcal{B}$ is empty, then $\mathcal{A}$ is empty.

The scheme *Sch3* deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary predicate $\mathcal{P}$, and states that:

$\overline{\overline{\{f; f \text{ ranges over functions from } \mathcal{A} \text{ into } \mathcal{B} : \mathcal{P}[f, \mathcal{A}, \mathcal{B}]\}}} =$
$\overline{\overline{\{f; f \text{ ranges over functions from } \mathcal{C} \text{ into } \mathcal{D} : \mathcal{P}[f, \mathcal{C}, \mathcal{D}]}}$
$\wedge\ \mathrm{rng}(f{\upharpoonright}\mathcal{A}) \subseteq \mathcal{B}\ \wedge\ \bigwedge_x (x \in \mathcal{C} \setminus \mathcal{A}\ \Rightarrow\ f(x) = \mathcal{F}(x))\}$

provided the following requirements are met:

- For every x such that $x \in \mathcal{C} \setminus \mathcal{A}$ holds $\mathcal{F}(x) \in \mathcal{D}$,
- $\mathcal{A} \subseteq \mathcal{C}$ and $\mathcal{B} \subseteq \mathcal{D}$,
- If $\mathcal{B}$ is empty, then $\mathcal{A}$ is empty, and

- Let f be a function from $\mathcal{C}$ into $\mathcal{D}$. Suppose that for every x such that $x \in \mathcal{C} \setminus \mathcal{A}$ holds $\mathcal{F}(x) = f(x)$. Then $\mathcal{P}[f, \mathcal{C}, \mathcal{D}]$ if and only if $\mathcal{P}[f{\upharpoonright}\mathcal{A}, \mathcal{A}, \mathcal{B}]$.

The scheme $Sch4$ deals with sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and a ternary predicate $\mathcal{P}$, and states that:

$$\overline{\overline{\{f; f \text{ ranges over functions from } \mathcal{A} \text{ into } \mathcal{B} : \mathcal{P}[f, \mathcal{A}, \mathcal{B}]\}}} =$$
$$\overline{\overline{\{f; f \text{ ranges over functions from } \mathcal{A} \cup \{\mathcal{C}\} \text{ into } \mathcal{B} \cup \{\mathcal{D}\} :}}$$
$$\overline{\mathcal{P}[f, \mathcal{A} \cup \{\mathcal{C}\}, \mathcal{B} \cup \{\mathcal{D}\}] \ \wedge \ \mathrm{rng}(f{\upharpoonright}\mathcal{A}) \subseteq \mathcal{B} \ \wedge \ f(\mathcal{C}) = \mathcal{D}\}}$$

provided the parameters meet the following conditions:

- If $\mathcal{B}$ is empty, then $\mathcal{A}$ is empty,
- $\mathcal{C} \notin \mathcal{A}$, and
- For every function f from $\mathcal{A} \cup \{\mathcal{C}\}$ into $\mathcal{B} \cup \{\mathcal{D}\}$ such that $f(\mathcal{C}) = \mathcal{D}$ holds $\mathcal{P}[f, \mathcal{A} \cup \{\mathcal{C}\}, \mathcal{B} \cup \{\mathcal{D}\}]$ iff $\mathcal{P}[f{\upharpoonright}\mathcal{A}, \mathcal{A}, \mathcal{B}]$.

We now state several propositions:

(34) For every function f from $n + 1$ into $k + 1$ such that f is onto and increasing and $f^{-1}(\{f(n)\}) = \{n\}$ holds $f(n) = k$.

(35) For every function f from $n+1$ into k such that $k \neq 0$ and $f^{-1}(\{f(n)\}) \neq \{n\}$ there exists m such that $m \in f^{-1}(\{f(n)\})$ and $m \neq n$.

(36) Let f be a function from n into k and g be a function from $n + m$ into $k + l$. Suppose g is increasing and $f = g{\upharpoonright}n$. Let given i, j. If $i \in \mathrm{rng}\, f$ and $j \in \mathrm{rng}\, f$ and $i < j$, then $\min^*(f^{-1}(\{i\})) < \min^*(f^{-1}(\{j\}))$.

(37) Let f be a function from $n+1$ into $k+1$. Suppose f is onto and increasing and $f^{-1}(\{f(n)\}) = \{n\}$. Then $\mathrm{rng}(f{\upharpoonright}n) \subseteq k$ and for every function g from n into k such that $g = f{\upharpoonright}n$ holds g is onto and increasing.

(38) Let f be a function from $n + 1$ into k and g be a function from n into k. Suppose f is onto and increasing and $f^{-1}(\{f(n)\}) \neq \{n\}$ and $f{\upharpoonright}n = g$. Then g is onto and increasing.

(39) Let f be a function from n into k and g be a function from $n + 1$ into $k + m$. Suppose f is onto and increasing and $f = g{\upharpoonright}n$. Let given i, j. If $i \in \mathrm{rng}\, g$ and $j \in \mathrm{rng}\, g$ and $i < j$, then $\min^*(g^{-1}(\{i\})) < \min^*(g^{-1}(\{j\}))$.

(40) Let f be a function from n into k and g be a function from $n + 1$ into $k + 1$. Suppose f is onto and increasing and $f = g{\upharpoonright}n$ and $g(n) = k$. Then g is onto and increasing and $g^{-1}(\{g(n)\}) = \{n\}$.

(41) Let f be a function from n into k and g be a function from $n + 1$ into k. Suppose f is onto and increasing and $f = g{\upharpoonright}n$ and $g(n) < k$. Then g is onto and increasing and $g^{-1}(\{g(n)\}) \neq \{n\}$.

In the sequel f_1 denotes a function from $n + 1$ into $k + 1$ and f denotes a function from n into k.

We now state the proposition

(42) $\overline{\overline{\{f_1 : f_1 \text{ is onto and increasing } \wedge \ f_1^{-1}(\{f_1(n)\}) = \{n\}\}}} =$

$$\overline{\overline{\{f : f \text{ is onto and increasing}\}}}.$$

In the sequel f' is a function from $n + 1$ into k.

The following proposition is true

(43)   For every l such that $l < k$ holds

$$\overline{\overline{\{f' : f' \text{ is onto and increasing } \wedge \ f'^{-1}(\{f'(n)\}) \neq \{n\} \ \wedge \ f'(n) = l\}}} =$$
$$\overline{\overline{\{f : f \text{ is onto and increasing}\}}}.$$

For simplicity, we adopt the following convention: D denotes a non empty set, F, G denote finite 0-sequences of D, F_1 denotes a finite 0-sequence of $\mathbb{N}$, b denotes a binary operation on D, and d, d_1, d_2 denote elements of D.

Let us consider D, F, b. Let us assume that b has a unity or $\operatorname{len} F \geq 1$. The functor $b \odot F$ yielding an element of D is defined as follows:

(Def. 3)(i) $b \odot F = \mathbf{1}_b$ if b has a unity and $\operatorname{len} F = 0$,

(ii)   there exists a function f from $\mathbb{N}$ into D such that $f(0) = F(0)$ and for every n such that $n + 1 < \operatorname{len} F$ holds $f(n + 1) = b(f(n), F(n + 1))$ and $b \odot F = f(\operatorname{len} F - 1)$, otherwise.

One can prove the following three propositions:

(44) $b \odot \langle d \rangle = d$.

(45)   If b has a unity or $\operatorname{len} F > 0$, then $b \odot F \frown \langle d \rangle = b(b \odot F, d)$.

(46)   If $F \neq \langle \rangle_D$, then there exist G, d such that $F = G \frown \langle d \rangle$.

The scheme $Sch5$ deals with a non empty set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every finite 0-sequence F of $\mathcal{A}$ holds $\mathcal{P}[F]$

provided the parameters satisfy the following conditions:

- $\mathcal{P}[\langle \rangle_{\mathcal{A}}]$, and
- For every finite 0-sequence F of $\mathcal{A}$ and for every element d of $\mathcal{A}$ such that $\mathcal{P}[F]$ holds $\mathcal{P}[F \frown \langle d \rangle]$.

Next we state the proposition

(47)   If b is associative and if b has a unity or $\operatorname{len} F \geq 1$ and $\operatorname{len} G \geq 1$, then $b \odot F \frown G = b(b \odot F, b \odot G)$.

Let us consider D and let us consider d, d_1. Then $\langle d, d_1 \rangle$ is a finite 0-sequence of D. Let us consider d_2. Then $\langle d, d_1, d_2 \rangle$ is a finite 0-sequence of D.

The following propositions are true:

(48) $b \odot \langle d_1, d_2 \rangle = b(d_1, d_2)$.

(49) $b \odot \langle d, d_1, d_2 \rangle = b(b(d, d_1), d_2)$.

Let us consider F_1. The functor $\sum F_1$ yields a natural number and is defined by:

(Def. 4) $\sum F_1 = +_{\mathbb{N}} \odot F_1$.

Let us consider F_1, x. Then $F_1(x)$ is a natural number.

One can prove the following propositions:

(50) If for every n such that $n \in \operatorname{dom} F_1$ holds $F_1(n) \leq k$, then $\sum F_1 \leq \operatorname{len} F_1 \cdot k$.

(51) If for every n such that $n \in \operatorname{dom} F_1$ holds $F_1(n) \geq k$, then $\sum F_1 \geq \operatorname{len} F_1 \cdot k$.

(52) If $\operatorname{len} F_1 > 0$ and there exists x such that $x \in \operatorname{dom} F_1$ and $F_1(x) = k$, then $\sum F_1 \geq k$.

(53) $\sum F_1 = 0$ iff $\operatorname{len} F_1 = 0$ or for every n such that $n \in \operatorname{dom} F_1$ holds $F_1(n) = 0$.

(54) For every function f and for every n holds $\bigcup \operatorname{rng}(f{\restriction}n) \cup f(n) = \bigcup \operatorname{rng}(f{\restriction}(n+1))$.

Now we present three schemes. The scheme *Sch6* deals with a non empty set $\mathcal{A}$, a natural number $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

> There exists a finite 0-sequence p of $\mathcal{A}$ such that $\operatorname{dom} p = \mathcal{B}$ and for every k such that $k \in \mathcal{B}$ holds $\mathcal{P}[k, p(k)]$

provided the parameters have the following property:

- For every k such that $k \in \mathcal{B}$ there exists an element x of $\mathcal{A}$ such that $\mathcal{P}[k, x]$.

The scheme *Sch7* deals with a non empty set $\mathcal{A}$ and a finite 0-sequence $\mathcal{B}$ of $\mathcal{A}$, and states that:

> There exists a finite 0-sequence C_1 of $\mathbb{N}$ such that $\operatorname{dom} C_1 = \operatorname{dom} \mathcal{B}$ and for every i such that $i \in \operatorname{dom} C_1$ holds $C_1(i) = \overline{\overline{\mathcal{B}(i)}}$ and $\overline{\overline{\bigcup \operatorname{rng} \mathcal{B}}} = \sum C_1$

provided the following requirements are met:

- For every i such that $i \in \operatorname{dom} \mathcal{B}$ holds $\mathcal{B}(i)$ is finite, and
- For all i, j such that $i \in \operatorname{dom} \mathcal{B}$ and $j \in \operatorname{dom} \mathcal{B}$ and $i \neq j$ holds $\mathcal{B}(i)$ misses $\mathcal{B}(j)$.

The scheme *Sch8* deals with finite sets $\mathcal{A}$, $\mathcal{B}$, a set $\mathcal{C}$, a function $\mathcal{D}$ from $\operatorname{card} \mathcal{B}$ into $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

> There exists a finite 0-sequence F of $\mathbb{N}$ such that
>
> (i) $\operatorname{dom} F = \operatorname{card} \mathcal{B}$,
>
> (ii) $\overline{\overline{\{g; g \text{ ranges over functions from } \mathcal{A} \text{ into } \mathcal{B} : \mathcal{P}[g]\}}} = \sum F$, and
>
> (iii) for every i such that $i \in \operatorname{dom} F$ holds $F(i) = \overline{\overline{\{g; g \text{ ranges over functions from } \mathcal{A} \text{ into } \mathcal{B} : \mathcal{P}[g] \wedge g(\mathcal{C}) = \mathcal{D}(i)\}}}$

provided the parameters have the following properties:

- $\mathcal{D}$ is onto and one-to-one,
- $\mathcal{B}$ is non empty, and
- $\mathcal{C} \in \mathcal{A}$.

One can prove the following propositions:

(55) $k \cdot (n \operatorname{block} k) = \overline{\overline{\{f' : f' \text{ is onto and increasing} \wedge f'^{-1}(\{f'(n)\}) \neq \{n\}\}}}$.

(56) $(n+1)\,\text{block}(k+1) = (k+1) \cdot (n\,\text{block}(k+1)) + (n\,\text{block}\,k)$.

(57) If $n \geq 1$, then $n\,\text{block}\,2 = \frac{1}{2} \cdot (2^n - 2)$.

(58) If $n \geq 2$, then $n\,\text{block}\,3 = \frac{1}{6} \cdot ((3^n - 3 \cdot 2^n) + 3)$.

(59) If $n \geq 3$, then $n\,\text{block}\,4 = \frac{1}{24} \cdot (((4^n - 4 \cdot 3^n) + 6 \cdot 2^n) - 4)$.

(60) $3! = 6$ and $4! = 24$.

(61) $\binom{n}{1} = n$ and $\binom{n}{2} = \frac{n \cdot (n-1)}{2}$ and $\binom{n}{3} = \frac{n \cdot (n-1) \cdot (n-2)}{6}$ and $\binom{n}{4} = \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3)}{24}$.

(62) $(n+1)\,\text{block}\,n = \binom{n+1}{2}$.

(63) $(n+2)\,\text{block}\,n = 3 \cdot \binom{n+2}{4} + \binom{n+2}{3}$.

(64) For every function F and for every y holds $\text{rng}(F{\upharpoonright}(\text{dom}\,F \setminus F^{-1}(\{y\}))) = \text{rng}\,F \setminus \{y\}$ and for every x such that $x \neq y$ holds $(F{\upharpoonright}(\text{dom}\,F \setminus F^{-1}(\{y\})))^{-1}(\{x\}) = F^{-1}(\{x\})$.

(65) If $\overline{\overline{X}} = k+1$ and $x \in X$, then $\overline{\overline{X \setminus \{x\}}} = k$.

The scheme $Sch9$ concerns a unary predicate $\mathcal{P}$ and a binary predicate $\mathcal{Q}$, and states that:

For every function F such that $\text{rng}\,F$ is finite holds $\mathcal{P}[F]$

provided the following conditions are met:

- $\mathcal{P}[\emptyset]$, and
- For every function F such that for every x such that $x \in \text{rng}\,F$ and $\mathcal{Q}[x, F]$ holds $\mathcal{P}[F{\upharpoonright}(\text{dom}\,F \setminus F^{-1}(\{x\}))]$ holds $\mathcal{P}[F]$.

We now state several propositions:

(66) For every subset N of $\mathbb{N}$ such that N is finite there exists k such that for every n such that $n \in N$ holds $n \leq k$.

(67) Let given X, Y, x, y. Suppose if Y is empty, then X is empty and $x \notin X$. Let F be a function from X into Y. Then there exists a function G from $X \cup \{x\}$ into $Y \cup \{y\}$ such that $G{\upharpoonright}X = F$ and $G(x) = y$.

(68) Let given X, Y, x, y such that if Y is empty, then X is empty. Let F be a function from X into Y and G be a function from $X \cup \{x\}$ into $Y \cup \{y\}$ such that $G{\upharpoonright}X = F$ and $G(x) = y$. Then

 (i)   if F is onto, then G is onto, and

 (ii) if $y \notin Y$ and F is one-to-one, then G is one-to-one.

(69) Let N be a finite subset of $\mathbb{N}$. Then there exists a function O_1 from N into $\text{card}\,N$ such that O_1 is bijective and for all n, k such that $n \in \text{dom}\,O_1$ and $k \in \text{dom}\,O_1$ and $n < k$ holds $O_1(n) < O_1(k)$.

(70) Let X, Y be finite sets and F be a function from X into Y. If $\text{card}\,X = \text{card}\,Y$, then F is onto iff F is one-to-one.

(71) Let F, G be functions and given y. Suppose $y \in \text{rng}(G \cdot F)$ and G is one-to-one. Then there exists x such that $x \in \text{dom}\,G$ and $x \in \text{rng}\,F$ and $G^{-1}(\{y\}) = \{x\}$ and $F^{-1}(\{x\}) = (G \cdot F)^{-1}(\{y\})$.

Let us consider N_1, K_1 and let f be a function from N_1 into K_1. We say that f is increasing if and only if:

(Def. 5) For all l, m such that $l \in \operatorname{rng} f$ and $m \in \operatorname{rng} f$ and $l < m$ holds $\min^*(f^{-1}(\{l\})) < \min^*(f^{-1}(\{m\}))$.

The following four propositions are true:

(72) For every function F from N_1 into K_1 such that F is increasing holds $\min^* \operatorname{rng} F = F(\min^* \operatorname{dom} F)$.

(73) Let F be a function from N_1 into K_1. Suppose $\operatorname{rng} F$ is finite. Then there exists a function I from N_1 into K_1 and there exists a permutation P of $\operatorname{rng} F$ such that $F = P \cdot I$ and $\operatorname{rng} F = \operatorname{rng} I$ and I is increasing.

(74) Let F be a function from N_1 into K_1. Suppose $\operatorname{rng} F$ is finite. Let I_1, I_2 be functions from N_1 into M_1 and P_1, P_2 be functions. Suppose that P_1 is one-to-one and P_2 is one-to-one and $\operatorname{rng} I_1 = \operatorname{rng} I_2$ and $\operatorname{rng} I_1 = \operatorname{dom} P_1$ and $\operatorname{dom} P_1 = \operatorname{dom} P_2$ and $F = P_1 \cdot I_1$ and $F = P_2 \cdot I_2$ and I_1 is increasing and I_2 is increasing. Then $P_1 = P_2$ and $I_1 = I_2$.

(75) Let F be a function from N_1 into K_1. Suppose $\operatorname{rng} F$ is finite. Let I_1, I_2 be functions from N_1 into K_1 and P_1, P_2 be permutations of $\operatorname{rng} F$. Suppose $F = P_1 \cdot I_1$ and $F = P_2 \cdot I_2$ and $\operatorname{rng} F = \operatorname{rng} I_1$ and $\operatorname{rng} F = \operatorname{rng} I_2$ and I_1 is increasing and I_2 is increasing. Then $P_1 = P_2$ and $I_1 = I_2$.

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