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The Banach Algebra of Bounded Linear Operators

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Summary. In this article, the basic properties of Banach algebra are described. This algebra is defined as the set of all bounded linear operators from one normed space to another.

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The papers [21], [8], [23], [25], [24], [5], [7], [6], [19], [4], [1], [2], [18], [10], [22], [13], [3], [20], [16], [15], [9], [12], [11], [14], and [17] provide the terminology and notation for this paper.

Let X be a non empty set and let f, g be elements of X^X. Then $g \cdot f$ is an element of X^X.

One can prove the following propositions:

(1) Let X, Y, Z be real linear spaces, f be a linear operator from X into Y, and g be a linear operator from Y into Z. Then $g \cdot f$ is a linear operator from X into Z.

(2) Let X, Y, Z be real normed spaces, f be a bounded linear operator from X into Y, and g be a bounded linear operator from Y into Z. Then

(i) $g \cdot f$ is a bounded linear operator from X into Z, and

(ii) for every vector x of X holds $\|(g \cdot f)(x)\| \leqslant (\mathrm{BdLinOpsNorm}(Y, Z))(g) \cdot (\mathrm{BdLinOpsNorm}(X, Y))(f) \cdot \|x\|$ and $(\mathrm{BdLinOpsNorm}(X, Z))(g \cdot f) \leqslant (\mathrm{BdLinOpsNorm}(Y, Z))(g) \cdot (\mathrm{BdLinOpsNorm}(X, Y))(f)$.

Let X be a real normed space and let f, g be bounded linear operators from X into X. Then $g \cdot f$ is a bounded linear operator from X into X.

Let X be a real normed space and let f, g be elements of $\mathrm{BdLinOps}(X, X)$. The functor $f+g$ yields an element of $\mathrm{BdLinOps}(X, X)$ and is defined as follows:

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(Def. 1) $f + g = (\mathrm{Add\_}(\mathrm{BdLinOps}(X, X),$
$\mathrm{RVectorSpaceOfLinearOperators}(X, X)))(f, g)$.

Let X be a real normed space and let f, g be elements of BdLinOps(X, X). The functor $g \cdot f$ yielding an element of BdLinOps(X, X) is defined as follows:

(Def. 2) $g \cdot f = \mathrm{modetrans}(g, X, X) \cdot \mathrm{modetrans}(f, X, X)$.

Let X be a real normed space, let f be an element of BdLinOps(X, X), and let a be a real number. The functor $a \cdot f$ yields an element of BdLinOps(X, X) and is defined by:

(Def. 3) $a \cdot f = (\mathrm{Mult\_}(\mathrm{BdLinOps}(X, X),$
$\mathrm{RVectorSpaceOfLinearOperators}(X, X)))(a, f)$.

Let X be a real normed space. The functor FuncMult(X) yielding a binary operation on BdLinOps(X, X) is defined as follows:

(Def. 4) For all elements f, g of BdLinOps(X, X) holds $(\mathrm{FuncMult}(X))(f, g) = f \cdot g$.

The following proposition is true

(3) For every real normed space X holds $\mathrm{id}_{\text{the carrier of } X}$ is a bounded linear operator from X into X.

Let X be a real normed space. The functor FuncUnit(X) yields an element of BdLinOps(X, X) and is defined as follows:

(Def. 5) $\mathrm{FuncUnit}(X) = \mathrm{id}_{\text{the carrier of } X}$.

One can prove the following propositions:

(4) Let X be a real normed space and f, g, h be bounded linear operators from X into X. Then $h = f \cdot g$ if and only if for every vector x of X holds $h(x) = f(g(x))$.

(5) For every real normed space X and for all bounded linear operators f, g, h from X into X holds $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.

(6) Let X be a real normed space and f be a bounded linear operator from X into X. Then $f \cdot \mathrm{id}_{\text{the carrier of } X} = f$ and $\mathrm{id}_{\text{the carrier of } X} \cdot f = f$.

(7) For every real normed space X and for all elements f, g, h of BdLinOps(X, X) holds $f \cdot (g \cdot h) = (f \cdot g) \cdot h$.

(8) For every real normed space X and for every element f of BdLinOps(X, X) holds $f \cdot \mathrm{FuncUnit}(X) = f$ and $\mathrm{FuncUnit}(X) \cdot f = f$.

(9) For every real normed space X and for all elements f, g, h of BdLinOps(X, X) holds $f \cdot (g + h) = f \cdot g + f \cdot h$.

(10) For every real normed space X and for all elements f, g, h of BdLinOps(X, X) holds $(g + h) \cdot f = g \cdot f + h \cdot f$.

(11) Let X be a real normed space, f, g be elements of BdLinOps(X, X), and a, b be real numbers. Then $(a \cdot b) \cdot (f \cdot g) = a \cdot f \cdot (b \cdot g)$.

(12) For every real normed space X and for all elements f, g of BdLinOps(X, X) and for every real number a holds $a \cdot (f \cdot g) = (a \cdot f) \cdot g$.

Let X be a real normed space. The functor RingOfBoundedLinearOperators(X) yielding a double loop structure is defined as follows:

(Def. 6) RingOfBoundedLinearOperators$(X) = \langle$BdLinOps(X, X), Add_(BdLinOps(X, X), RVectorSpaceOfLinearOperators(X, X)), FuncMult(X), FuncUnit(X), Zero_(BdLinOps(X, X), RVectorSpaceOfLinearOperators(X, X))$\rangle$.

Let X be a real normed space. Observe that RingOfBoundedLinearOperators(X) is non empty and strict.

One can prove the following propositions:

(13) Let X be a real normed space and x, y, z be elements of RingOfBoundedLinearOperators(X). Then $x + y = y + x$ and $(x + y) + z = x + (y + z)$ and $x + 0_{\mathrm{RingOfBoundedLinearOperators}(X)} = x$ and there exists an element t of RingOfBoundedLinearOperators(X) such that $x + t = 0_{\mathrm{RingOfBoundedLinearOperators}(X)}$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot \mathbf{1}_{\mathrm{RingOfBoundedLinearOperators}(X)} = x$ and $\mathbf{1}_{\mathrm{RingOfBoundedLinearOperators}(X)} \cdot x = x$ and $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y + z) \cdot x = y \cdot x + z \cdot x$.

(14) For every real normed space X holds RingOfBoundedLinearOperators(X) is a ring.

Let X be a real normed space. Note that RingOfBoundedLinearOperators(X) is Abelian, add-associative, right zeroed, right complementable, associative, left unital, right unital, and distributive.

Let X be a real normed space.

The functor RAlgebraOfBoundedLinearOperators(X) yielding an algebra structure is defined as follows:

(Def. 7) RAlgebraOfBoundedLinearOperators$(X) = \langle$BdLinOps(X, X), FuncMult(X), Add_(BdLinOps(X, X), RVectorSpaceOfLinearOperators(X, X)), Mult_(BdLinOps(X, X), RVectorSpaceOfLinearOperators(X, X)), FuncUnit(X), Zero_(BdLinOps(X, X), RVectorSpaceOfLinearOperators(X, X))$\rangle$.

Let X be a real normed space.

Observe that RAlgebraOfBoundedLinearOperators(X) is non empty and strict.

Next we state the proposition

(15) Let X be a real normed space, x, y, z be elements of RAlgebraOfBoundedLinearOperators(X), and a, b be real numbers. Then $x + y = y + x$ and $(x + y) + z = x + (y + z)$ and $x + 0_{\mathrm{RAlgebraOfBoundedLinearOperators}(X)} = x$ and there exists an element t of RAlgebraOfBoundedLinearOperators(X) such that $x + t = 0_{\mathrm{RAlgebraOfBoundedLinearOperators}(X)}$ and $(x \cdot y) \cdot z =$

$x \cdot (y \cdot z)$ and $x \cdot \mathbf{1}_{\mathrm{RAlgebraOfBoundedLinearOperators}(X)} = x$ and $\mathbf{1}_{\mathrm{RAlgebraOfBoundedLinearOperators}(X)} \cdot x = x$ and $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(y+z) \cdot x = y \cdot x + z \cdot x$ and $a \cdot (x \cdot y) = (a \cdot x) \cdot y$ and $a \cdot (x+y) = a \cdot x + a \cdot y$ and $(a+b) \cdot x = a \cdot x + b \cdot x$ and $(a \cdot b) \cdot x = a \cdot (b \cdot x)$ and $(a \cdot b) \cdot (x \cdot y) = a \cdot x \cdot (b \cdot y)$.

A BL algebra is an Abelian add-associative right zeroed right complementable associative algebra-like non empty algebra structure.

The following proposition is true

(16) For every real normed space X holds RAlgebraOfBoundedLinearOperators(X) is a BL algebra.

One can check that l1-Space is complete.

Let us mention that l1-Space is non trivial.

One can verify that there exists a real Banach space which is non trivial.

One can prove the following propositions:

(17) For every non trivial real normed space X there exists a vector w of X such that $\|w\| = 1$.

(18) For every non trivial real normed space X holds (BdLinOpsNorm(X, X)) $(\mathrm{id}_{\text{the carrier of } X}) = 1$.

We introduce normed algebra structures which are extensions of algebra structure and normed structure and are systems

⟨ a carrier, a multiplication, an addition, an external multiplication, a unity, a zero, a norm ⟩,

where the carrier is a set, the multiplication and the addition are binary operations on the carrier, the external multiplication is a function from [: ℝ, the carrier :] into the carrier, the unity and the zero are elements of the carrier, and the norm is a function from the carrier into ℝ.

Let us mention that there exists a normed algebra structure which is non empty.

Let X be a real normed space.

The functor RNormedAlgebraOfBoundedLinearOperators(X) yields a normed algebra structure and is defined by:

(Def. 8) RNormedAlgebraOfBoundedLinearOperators(X) = ⟨BdLinOps(X, X), FuncMult(X), Add_(BdLinOps(X, X), RVectorSpaceOfLinearOperators (X, X)), Mult_(BdLinOps(X, X), RVectorSpaceOfLinearOperators(X, X)), FuncUnit(X), Zero_(BdLinOps(X, X), RVectorSpaceOfLinearOperators (X, X)), BdLinOpsNorm(X, X)⟩.

Let X be a real normed space. One can verify that RNormedAlgebraOfBoundedLinearOperators(X) is non empty and strict.

Next we state two propositions:

(19) Let X be a real normed space, x, y, z be elements of RNormedAlgebraOfBoundedLinearOperators(X), and a, b be real num-

bers. Then $x + y = y + x$ and $(x + y) + z = x + (y + z)$ and $x + 0_{\mathrm{RNormedAlgebraOfBoundedLinearOperators}(X)} = x$ and there exists an element t of RNormedAlgebraOfBoundedLinearOperators(X) such that $x + t = 0_{\mathrm{RNormedAlgebraOfBoundedLinearOperators}(X)}$ and $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ and $x \cdot \mathbf{1}_{\mathrm{RNormedAlgebraOfBoundedLinearOperators}(X)} = x$ and $\mathbf{1}_{\mathrm{RNormedAlgebraOfBoundedLinearOperators}(X)} \cdot x = x$ and $x \cdot (y+z) = x \cdot y + x \cdot z$ and $(y+z) \cdot x = y \cdot x + z \cdot x$ and $a \cdot (x \cdot y) = (a \cdot x) \cdot y$ and $(a \cdot b) \cdot (x \cdot y) = a \cdot x \cdot (b \cdot y)$ and $a \cdot (x+y) = a \cdot x + a \cdot y$ and $(a+b) \cdot x = a \cdot x + b \cdot x$ and $(a \cdot b) \cdot x = a \cdot (b \cdot x)$ and $1 \cdot x = x$.

(20) Let X be a real normed space.
Then RNormedAlgebraOfBoundedLinearOperators(X) is real normed space-like, Abelian, add-associative, right zeroed, right complementable, associative, algebra-like, and real linear space-like.

Let us observe that there exists a non empty normed algebra structure which is real normed space-like, Abelian, add-associative, right zeroed, right complementable, associative, algebra-like, real linear space-like, and strict.

A normed algebra is a real normed space-like Abelian add-associative right zeroed right complementable associative algebra-like real linear space-like non empty normed algebra structure.

Let X be a real normed space.

Observe that RNormedAlgebraOfBoundedLinearOperators(X) is real normed space-like, Abelian, add-associative, right zeroed, right complementable, associative, algebra-like, and real linear space-like.

Let X be a non empty normed algebra structure. We say that X is Banach Algebra-like1 if and only if:

(Def. 9) For all elements x, y of X holds $\|x \cdot y\| \leqslant \|x\| \cdot \|y\|$.

We say that X is Banach Algebra-like2 if and only if:

(Def. 10) $\|\mathbf{1}_X\| = 1$.

We say that X is Banach Algebra-like3 if and only if:

(Def. 11) For every real number a and for all elements x, y of X holds $a \cdot (x \cdot y) = x \cdot (a \cdot y)$.

Let X be a normed algebra. We say that X is Banach Algebra-like if and only if the condition (Def. 12) is satisfied.

(Def. 12) X is Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, left unital, left distributive, and complete.

Let us mention that every normed algebra which is Banach Algebra-like is also Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, left distributive, left unital, and complete and every normed algebra which is Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, left distributive, left unital, and complete is also Banach Algebra-like.

Let X be a non trivial real Banach space.

Note that RNormedAlgebraOfBoundedLinearOperators(X) is Banach Algebra-like.

One can verify that there exists a normed algebra which is Banach Algebra-like.

A Banach algebra is a Banach Algebra-like normed algebra.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Grzegorz Bancerek. Sequences of ordinal numbers. *Formalized Mathematics*, 1(**2**):281–290, 1990.
[3] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.
[4] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[8] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[9] Noboru Endou, Yasumasa Suzuki, and Yasunari Shidama. Real linear space of real sequences. *Formalized Mathematics*, 11(**3**):249–253, 2003.
[10] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[11] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[12] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[13] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[14] Henryk Oryszczyszyn and Krzysztof Prażmowski. Real functions spaces. *Formalized Mathematics*, 1(**3**):555–561, 1990.
[15] Jan Popiołek. Introduction to Banach and Hilbert spaces - part I. *Formalized Mathematics*, 2(**4**):511–516, 1991.
[16] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[17] Yasunari Shidama. Banach space of bounded linear operators. *Formalized Mathematics*, 12(**1**):39–48, 2003.
[18] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics*.
[19] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[20] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[22] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[23] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[25] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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