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Basic Properties of Rough Sets and Rough Membership Function[1]

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Summary. We present basic concepts concerning rough set theory. We define tolerance and approximation spaces and rough membership function. Different rough inclusions as well as the predicate of rough equality of sets are also introduced.

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The notation and terminology used here are introduced in the following papers: [21], [9], [25], [19], [1], [13], [22], [11], [20], [26], [28], [6], [2], [10], [5], [27], [8], [3], [15], [14], [7], [4], [16], [23], [24], [17], [18], and [12].

1. Preliminaries

Let A be a set. One can verify that $\langle A, \mathrm{id}_A \rangle$ is discrete.

The following proposition is true

(1) For every set X such that $\nabla_X \subseteq \mathrm{id}_X$ holds X is trivial.

Let A be a relational structure. We say that A is diagonal if and only if:

(Def. 1) The internal relation of $A \subseteq \mathrm{id}_{\text{the carrier of } A}$.

Let A be a non trivial set. Observe that $\langle A, \nabla_A \rangle$ is non diagonal.

We now state the proposition

(2) For every reflexive relational structure L holds $\mathrm{id}_{\text{the carrier of } L} \subseteq$ the internal relation of L.

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Let us note that every reflexive relational structure which is non discrete is also non trivial and every relational structure which is reflexive and trivial is also discrete.

One can prove the following proposition

(3) For every set X and for every total reflexive binary relation R on X holds $\mathrm{id}_X \subseteq R$.

One can verify that every relational structure which is discrete is also diagonal and every relational structure which is non diagonal is also non discrete.

One can verify that there exists a relational structure which is non diagonal and non empty.

We now state three propositions:

(4) Let A be a non diagonal non empty relational structure. Then there exist elements x, y of A such that $x \neq y$ and $\langle x, y \rangle \in$ the internal relation of A.

(5) For every set D and for all finite sequences p, q of elements of D holds $\bigcup (p \frown q) = \bigcup p \cup \bigcup q$.

(6) For all functions p, q such that q is disjoint valued and $p \subseteq q$ holds p is disjoint valued.

One can verify that every function which is empty is also disjoint valued.

Let A be a set. One can verify that there exists a finite sequence of elements of A which is disjoint valued.

Let A be a non empty set. Observe that there exists a finite sequence of elements of A which is non empty and disjoint valued.

Let A be a set, let X be a finite sequence of elements of 2^A, and let n be a natural number. Then $X(n)$ is a subset of A.

Let A be a set and let X be a finite sequence of elements of 2^A. Then $\bigcup X$ is a subset of A.

Let A be a finite set and let R be a binary relation on A. One can check that $\langle A, R \rangle$ is finite.

One can prove the following proposition

(7) For all sets X, x, y and for every tolerance T of X such that $x \in [y]_T$ holds $y \in [x]_T$.

2. Tolerance and Approximation Spaces

Let P be a relational structure. We say that P has equivalence relation if and only if:

(Def. 2) The internal relation of P is an equivalence relation of the carrier of P.

We say that P has tolerance relation if and only if:

(Def. 3) The internal relation of P is a tolerance of the carrier of P.

Let us note that every relational structure which has equivalence relation has also tolerance relation.

Let A be a set. Observe that $\langle A, \mathrm{id}_A \rangle$ has equivalence relation.

One can verify that there exists a relational structure which is discrete, finite, and non empty and has equivalence relation and there exists a relational structure which is non diagonal, finite, and non empty and has equivalence relation.

An approximation space is a non empty relational structure with equivalence relation. A tolerance space is a non empty relational structure with tolerance relation.

Let A be a tolerance space. Note that the internal relation of A is total, reflexive, and symmetric.

Let A be an approximation space. Observe that the internal relation of A is transitive.

Let A be a tolerance space and let X be a subset of A. The functor $\mathrm{LAp}(X)$ yielding a subset of A is defined as follows:

(Def. 4) $\mathrm{LAp}(X) = \{x; x$ ranges over elements of A: $[x]_{\text{the internal relation of } A} \subseteq X\}$.

The functor $\mathrm{UAp}(X)$ yielding a subset of A is defined as follows:

(Def. 5) $\mathrm{UAp}(X) = \{x; x$ ranges over elements of A: $[x]_{\text{the internal relation of } A}$ meets $X\}$.

Let A be a tolerance space and let X be a subset of A. The functor $\mathrm{BndAp}(X)$ yielding a subset of A is defined as follows:

(Def. 6) $\mathrm{BndAp}(X) = \mathrm{UAp}(X) \setminus \mathrm{LAp}(X)$.

Let A be a tolerance space and let X be a subset of A. We say that X is rough if and only if:

(Def. 7) $\mathrm{BndAp}(X) \neq \emptyset$.

We introduce X is exact as an antonym of X is rough.

In the sequel A is a tolerance space and X, Y are subsets of A.

Next we state a number of propositions:

(8) For every set x such that $x \in \mathrm{LAp}(X)$ holds $[x]_{\text{the internal relation of } A} \subseteq X$.

(9) For every element x of A such that $[x]_{\text{the internal relation of } A} \subseteq X$ holds $x \in \mathrm{LAp}(X)$.

(10) For every set x such that $x \in \mathrm{UAp}(X)$ holds $[x]_{\text{the internal relation of } A}$ meets X.

(11) For every element x of A such that $[x]_{\text{the internal relation of } A}$ meets X holds $x \in \mathrm{UAp}(X)$.

(12) $\mathrm{LAp}(X) \subseteq X$.

(13) $X \subseteq \mathrm{UAp}(X)$.

(14) $\mathrm{LAp}(X) \subseteq \mathrm{UAp}(X)$.

(15) X is exact iff $\mathrm{LAp}(X) = X$.

(16) X is exact iff $\mathrm{UAp}(X) = X$.

(17) $X = \mathrm{LAp}(X)$ iff $X = \mathrm{UAp}(X)$.

(18) $\mathrm{LAp}(\emptyset_A) = \emptyset$.

(19) $\mathrm{UAp}(\emptyset_A) = \emptyset$.

(20) $\mathrm{LAp}(\Omega_A) = \Omega_A$.

(21) $\mathrm{UAp}(\Omega_A) = \Omega_A$.

(22) $\mathrm{LAp}(X \cap Y) = \mathrm{LAp}(X) \cap \mathrm{LAp}(Y)$.

(23) $\mathrm{UAp}(X \cup Y) = \mathrm{UAp}(X) \cup \mathrm{UAp}(Y)$.

(24) If $X \subseteq Y$, then $\mathrm{LAp}(X) \subseteq \mathrm{LAp}(Y)$.

(25) If $X \subseteq Y$, then $\mathrm{UAp}(X) \subseteq \mathrm{UAp}(Y)$.

(26) $\mathrm{LAp}(X) \cup \mathrm{LAp}(Y) \subseteq \mathrm{LAp}(X \cup Y)$.

(27) $\mathrm{UAp}(X \cap Y) \subseteq \mathrm{UAp}(X) \cap \mathrm{UAp}(Y)$.

(28) $\mathrm{LAp}(X^{\mathrm{c}}) = (\mathrm{UAp}(X))^{\mathrm{c}}$.

(29) $\mathrm{UAp}(X^{\mathrm{c}}) = (\mathrm{LAp}(X))^{\mathrm{c}}$.

(30) $\mathrm{UAp}(\mathrm{LAp}(\mathrm{UAp}(X))) = \mathrm{UAp}(X)$.

(31) $\mathrm{LAp}(\mathrm{UAp}(\mathrm{LAp}(X))) = \mathrm{LAp}(X)$.

(32) $\mathrm{BndAp}(X) = \mathrm{BndAp}(X^{\mathrm{c}})$.

In the sequel A is an approximation space and X is a subset of A.

The following four propositions are true:

(33) $\mathrm{LAp}(\mathrm{LAp}(X)) = \mathrm{LAp}(X)$.

(34) $\mathrm{LAp}(\mathrm{LAp}(X)) = \mathrm{UAp}(\mathrm{LAp}(X))$.

(35) $\mathrm{UAp}(\mathrm{UAp}(X)) = \mathrm{UAp}(X)$.

(36) $\mathrm{UAp}(\mathrm{UAp}(X)) = \mathrm{LAp}(\mathrm{UAp}(X))$.

Let A be an approximation space. Note that there exists a subset of A which is exact.

Let A be an approximation space and let X be a subset of A. One can check that $\mathrm{LAp}(X)$ is exact and $\mathrm{UAp}(X)$ is exact.

The following proposition is true

(37) Let A be an approximation space, X be a subset of A, and x, y be sets. If $x \in \mathrm{UAp}(X)$ and $\langle x, y \rangle \in$ the internal relation of A, then $y \in \mathrm{UAp}(X)$.

Let A be a non diagonal approximation space. Observe that there exists a subset of A which is rough.

Let A be an approximation space and let X be a subset of A. Rough set of X is defined by:

(Def. 8) It $= \langle \mathrm{LAp}(X), \mathrm{UAp}(X) \rangle$.

3. Membership Function

Let A be a finite tolerance space and let x be an element of A. One can check that $\mathrm{card}([x]_{\text{the internal relation of } A})$ is non empty.

Let A be a finite tolerance space and let X be a subset of A. The functor MemberFunc(X, A) yielding a function from the carrier of A into $\mathbb{R}$ is defined by:

(Def. 9) For every element x of A holds $(\mathrm{MemberFunc}(X, A))(x) = \frac{\mathrm{card}(X \cap [x]_{\text{the internal relation of } A})}{\mathrm{card}([x]_{\text{the internal relation of } A})}$.

In the sequel A denotes a finite tolerance space, X denotes a subset of A, and x denotes an element of A.

One can prove the following propositions:

(38) $0 \leqslant (\mathrm{MemberFunc}(X, A))(x)$ and $(\mathrm{MemberFunc}(X, A))(x) \leqslant 1$.

(39) $(\mathrm{MemberFunc}(X, A))(x) \in [0, 1]$.

In the sequel A is a finite approximation space, X, Y are subsets of A, and x is an element of A.

We now state four propositions:

(40) $(\mathrm{MemberFunc}(X, A))(x) = 1$ iff $x \in \mathrm{LAp}(X)$.

(41) $(\mathrm{MemberFunc}(X, A))(x) = 0$ iff $x \in (\mathrm{UAp}(X))^{\mathrm{c}}$.

(42) $0 < (\mathrm{MemberFunc}(X, A))(x)$ and $(\mathrm{MemberFunc}(X, A))(x) < 1$ iff $x \in \mathrm{BndAp}(X)$.

(43) For every discrete approximation space A holds every subset of A is exact.

Let A be a discrete approximation space. Note that every subset of A is exact.

The following propositions are true:

(44) For every discrete finite approximation space A and for every subset X of A holds $\mathrm{MemberFunc}(X, A) = \chi_{X, \text{the carrier of } A}$.

(45) Let A be a finite approximation space, X be a subset of A, and x, y be sets. If $\langle x, y \rangle \in$ the internal relation of A, then $(\mathrm{MemberFunc}(X, A))(x) = (\mathrm{MemberFunc}(X, A))(y)$.

(46) $(\mathrm{MemberFunc}(X^{\mathrm{c}}, A))(x) = 1 - (\mathrm{MemberFunc}(X, A))(x)$.

(47) If $X \subseteq Y$, then $(\mathrm{MemberFunc}(X, A))(x) \leqslant (\mathrm{MemberFunc}(Y, A))(x)$.

(48) $(\mathrm{MemberFunc}(X \cup Y, A))(x) \geqslant (\mathrm{MemberFunc}(X, A))(x)$.

(49) $(\mathrm{MemberFunc}(X \cap Y, A))(x) \leqslant (\mathrm{MemberFunc}(X, A))(x)$.

(50) $(\mathrm{MemberFunc}(X \cup Y, A))(x) \geqslant \max((\mathrm{MemberFunc}(X, A))(x), (\mathrm{MemberFunc}(Y, A))(x))$.

(51) If X misses Y, then $(\mathrm{MemberFunc}(X \cup Y, A))(x) = (\mathrm{MemberFunc}(X, A))(x) + (\mathrm{MemberFunc}(Y, A))(x)$.

(52) $(\mathrm{MemberFunc}(X \cap Y, A))(x) \leqslant \min((\mathrm{MemberFunc}(X, A))(x),$ $(\mathrm{MemberFunc}(Y, A))(x))$.

Let A be a finite tolerance space, let X be a finite sequence of elements of $2^{\text{the carrier of } A}$, and let x be an element of A. The functor $\mathrm{FinSeqM}(x, X)$ yields a finite sequence of elements of $\mathbb{R}$ and is defined as follows:

(Def. 10) $\operatorname{dom} \mathrm{FinSeqM}(x, X) = \operatorname{dom} X$ and for every natural number n such that $n \in \operatorname{dom} X$ holds $(\mathrm{FinSeqM}(x, X))(n) = (\mathrm{MemberFunc}(X(n), A))(x)$.

We now state several propositions:

(53) Let X be a finite sequence of elements of $2^{\text{the carrier of } A}$, x be an element of A, and y be an element of $2^{\text{the carrier of } A}$. Then $\mathrm{FinSeqM}(x, X \frown \langle y \rangle) = (\mathrm{FinSeqM}(x, X)) \frown \langle (\mathrm{MemberFunc}(y, A))(x) \rangle$.

(54) $(\mathrm{MemberFunc}(\emptyset_A, A))(x) = 0$.

(55) For every disjoint valued finite sequence X of elements of $2^{\text{the carrier of } A}$ holds $(\mathrm{MemberFunc}(\bigcup X, A))(x) = \sum \mathrm{FinSeqM}(x, X)$.

(56) $\mathrm{LAp}(X) = \{x; x$ ranges over elements of A: $(\mathrm{MemberFunc}(X, A))(x) = 1\}$.

(57) $\mathrm{UAp}(X) = \{x; x$ ranges over elements of A: $(\mathrm{MemberFunc}(X, A))(x) > 0\}$.

(58) $\mathrm{BndAp}(X) = \{x; x$ ranges over elements of A: $0 < (\mathrm{MemberFunc}(X, A))(x) \wedge (\mathrm{MemberFunc}(X, A))(x) < 1\}$.

4. Rough Inclusion

In the sequel A is a tolerance space and X, Y, Z are subsets of A.

Let A be a tolerance space and let X, Y be subsets of A. The predicate $X \subseteq_* Y$ is defined as follows:

(Def. 11) $\mathrm{LAp}(X) \subseteq \mathrm{LAp}(Y)$.

The predicate $X \subseteq^* Y$ is defined as follows:

(Def. 12) $\mathrm{UAp}(X) \subseteq \mathrm{UAp}(Y)$.

Let A be a tolerance space and let X, Y be subsets of A. The predicate $X \subseteq^*_* Y$ is defined as follows:

(Def. 13) $X \subseteq_* Y$ and $X \subseteq^* Y$.

One can prove the following three propositions:

(59) If $X \subseteq_* Y$ and $Y \subseteq_* Z$, then $X \subseteq_* Z$.

(60) If $X \subseteq^* Y$ and $Y \subseteq^* Z$, then $X \subseteq^* Z$.

(61) If $X \subseteq^*_* Y$ and $Y \subseteq^*_* Z$, then $X \subseteq^*_* Z$.

5. Rough Equality of Sets

Let A be a tolerance space and let X, Y be subsets of A. The predicate $X =_* Y$ is defined by:

(Def. 14) $\mathrm{LAp}(X) = \mathrm{LAp}(Y)$.

Let us notice that the predicate $X =_* Y$ is reflexive and symmetric. The predicate $X =^* Y$ is defined as follows:

(Def. 15) $\mathrm{UAp}(X) = \mathrm{UAp}(Y)$.

Let us notice that the predicate $X =^* Y$ is reflexive and symmetric. The predicate $X =^*_* Y$ is defined by:

(Def. 16) $\mathrm{LAp}(X) = \mathrm{LAp}(Y)$ and $\mathrm{UAp}(X) = \mathrm{UAp}(Y)$.

Let us notice that the predicate $X =^*_* Y$ is reflexive and symmetric.

Let A be a tolerance space and let X, Y be subsets of A. Let us observe that $X =_* Y$ if and only if:

(Def. 17) $X \subseteq_* Y$ and $Y \subseteq_* X$.

Let us observe that $X =^* Y$ if and only if:

(Def. 18) $X \subseteq^* Y$ and $Y \subseteq^* X$.

Let us observe that $X =^*_* Y$ if and only if:

(Def. 19) $X =_* Y$ and $X =^* Y$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[3] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.
[4] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.
[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[9] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[10] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.
[11] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[12] Adam Grabowski. On the category of posets. *Formalized Mathematics*, 5(**4**):501–505, 1996.
[13] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[14] Andrzej Nędzusiak. Probability. *Formalized Mathematics*, 1(**4**):745–749, 1990.
[15] Andrzej Nędzusiak. σ-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.
[16] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.

[17] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.
[18] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[19] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[20] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[21] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[22] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[23] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[24] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[25] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[26] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[27] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[28] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(**1**):85–89, 1990.

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