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Term Orders

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Summary. We continue the formalization of [5] towards Gröbner Bases. Here we deal with term orders, that is with orders on bags. We introduce the notions of head term, head coefficient, and head monomial necessary to define reduction for polynomials.

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The papers [16], [21], [22], [1], [10], [23], [7], [8], [3], [2], [12], [20], [17], [4], [6], [9], [11], [24], [14], [13], [18], [19], and [15] provide the terminology and notation for this paper.

1. Preliminaries

One can check that there exists a loop structure which is non trivial.

Let us mention that there exists a non trivial loop structure which is add-associative, right complementable, and right zeroed.

Let X be a set and let b be a bag of X. We say that b is non-zero if and only if:

(Def. 1) $b \neq \mathrm{EmptyBag}\, X$.

Next we state two propositions:

(1) For every set X and for all bags b_1, b_2 of X holds $b_1 \mid b_2$ iff there exists a bag b of X such that $b_2 = b_1 + b$.

(2) Let n be an ordinal number, L be an add-associative right complementable right zeroed unital distributive non empty double loop structure, and p be a series of n, L. Then $0\_(n, L) * p = 0\_(n, L)$.

Let n be an ordinal number, let L be an add-associative right complementable right zeroed unital distributive non empty double loop structure, and let m_1, m_2 be monomials of n, L. Note that $m_1 * m_2$ is monomial-like.

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Let n be an ordinal number, let L be an add-associative right complementable right zeroed distributive non empty double loop structure, and let c_1, c_2 be constant polynomials of n, L. One can verify that $c_1 * c_2$ is constant.

One can prove the following two propositions:

(3) Let n be an ordinal number, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, b, b' be bags of n, and a, a' be non-zero elements of L. Then $\mathrm{Monom}(a \cdot a', b + b') = \mathrm{Monom}(a, b) * \mathrm{Monom}(a', b')$.

(4) Let n be an ordinal number, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, and a, a' be elements of L. Then $a \cdot a' \mathbin{\_}(n, L) = (a \mathbin{\_}(n, L)) * (a' \mathbin{\_}(n, L))$.

2. Term Orders

Let n be an ordinal number. One can verify that there exists a term order of n which is admissible and connected.

Let n be a natural number. Observe that every admissible term order of n is well founded.

Let n be an ordinal number, let T be a term order of n, and let b, b' be bags of n. The predicate $b \leqslant_T b'$ is defined by:

(Def. 2) $\langle b, b' \rangle \in T$.

Let n be an ordinal number, let T be a term order of n, and let b, b' be bags of n. The predicate $b <_T b'$ is defined by:

(Def. 3) $b \leqslant_T b'$ and $b \neq b'$.

Let n be an ordinal number, let T be a term order of n, and let b_1, b_2 be bags of n. The functor $\min_T(b_1, b_2)$ yields a bag of n and is defined as follows:

(Def. 4) $\min_T(b_1, b_2) = \begin{cases} b_1, & \text{if } b_1 \leqslant_T b_2, \\ b_2, & \text{otherwise.} \end{cases}$

The functor $\max_T(b_1, b_2)$ yields a bag of n and is defined as follows:

(Def. 5) $\max_T(b_1, b_2) = \begin{cases} b_1, & \text{if } b_2 \leqslant_T b_1, \\ b_2, & \text{otherwise.} \end{cases}$

We now state a number of propositions:

(5) Let n be an ordinal number, T be a connected term order of n, and b_1, b_2 be bags of n. Then $b_1 \leqslant_T b_2$ if and only if $b_2 \not<_T b_1$.

(6) For every ordinal number n and for every term order T of n and for every bag b of n holds $b \leqslant_T b$.

(7) Let n be an ordinal number, T be a term order of n, and b_1, b_2 be bags of n. If $b_1 \leqslant_T b_2$ and $b_2 \leqslant_T b_1$, then $b_1 = b_2$.

(8) Let n be an ordinal number, T be a term order of n, and b_1, b_2, b_3 be bags of n. If $b_1 \leqslant_T b_2$ and $b_2 \leqslant_T b_3$, then $b_1 \leqslant_T b_3$.

(9) For every ordinal number n and for every admissible term order T of n and for every bag b of n holds $\mathrm{EmptyBag}\, n \leqslant_T b$.

(10) Let n be an ordinal number, T be an admissible term order of n, and b_1, b_2 be bags of n. If $b_1 \mid b_2$, then $b_1 \leqslant_T b_2$.

(11) For every ordinal number n and for every term order T of n and for all bags b_1, b_2 of n holds $\min_T(b_1, b_2) = b_1$ or $\min_T(b_1, b_2) = b_2$.

(12) For every ordinal number n and for every term order T of n and for all bags b_1, b_2 of n holds $\max_T(b_1, b_2) = b_1$ or $\max_T(b_1, b_2) = b_2$.

(13) Let n be an ordinal number, T be a connected term order of n, and b_1, b_2 be bags of n. Then $\min_T(b_1, b_2) \leqslant_T b_1$ and $\min_T(b_1, b_2) \leqslant_T b_2$.

(14) Let n be an ordinal number, T be a connected term order of n, and b_1, b_2 be bags of n. Then $b_1 \leqslant_T \max_T(b_1, b_2)$ and $b_2 \leqslant_T \max_T(b_1, b_2)$.

(15) Let n be an ordinal number, T be a connected term order of n, and b_1, b_2 be bags of n. Then $\min_T(b_1, b_2) = \min_T(b_2, b_1)$ and $\max_T(b_1, b_2) = \max_T(b_2, b_1)$.

(16) Let n be an ordinal number, T be a connected term order of n, and b_1, b_2 be bags of n. Then $\min_T(b_1, b_2) = b_1$ if and only if $\max_T(b_1, b_2) = b_2$.

3. Head Terms, Head Monomials, and Head Coefficients

Let n be an ordinal number, let T be a connected term order of n, let L be a non empty zero structure, and let p be a polynomial of n, L. The functor $\mathrm{HT}(p, T)$ yields an element of $\mathrm{Bags}\, n$ and is defined as follows:

(Def. 6) $\mathrm{Support}\, p = \emptyset$ and $\mathrm{HT}(p, T) = \mathrm{EmptyBag}\, n$ or $\mathrm{HT}(p, T) \in \mathrm{Support}\, p$ and for every bag b of n such that $b \in \mathrm{Support}\, p$ holds $b \leqslant_T \mathrm{HT}(p, T)$.

Let n be an ordinal number, let T be a connected term order of n, let L be a non empty zero structure, and let p be a polynomial of n, L. The functor $\mathrm{HC}(p, T)$ yielding an element of L is defined as follows:

(Def. 7) $\mathrm{HC}(p, T) = p(\mathrm{HT}(p, T))$.

Let n be an ordinal number, let T be a connected term order of n, let L be a non empty zero structure, and let p be a polynomial of n, L. The functor $\mathrm{HM}(p, T)$ yielding a monomial of n, L is defined by:

(Def. 8) $\mathrm{HM}(p, T) = \mathrm{Monom}(\mathrm{HC}(p, T), \mathrm{HT}(p, T))$.

Let n be an ordinal number, let T be a connected term order of n, let L be a non trivial zero structure, and let p be a non-zero polynomial of n, L. Observe that $\mathrm{HM}(p, T)$ is non-zero and $\mathrm{HC}(p, T)$ is non-zero.

The following propositions are true:

(17) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, and p be a polynomial of n, L. Then $\mathrm{HC}(p,T) = 0_L$ if and only if $p = 0\_(n,L)$.

(18) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, and p be a polynomial of n, L. Then $(\mathrm{HM}(p,T))(\mathrm{HT}(p,T)) = p(\mathrm{HT}(p,T))$.

(19) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, p be a polynomial of n, L, and b be a bag of n. If $b \neq \mathrm{HT}(p,T)$, then $(\mathrm{HM}(p,T))(b) = 0_L$.

(20) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, and p be a polynomial of n, L. Then $\mathrm{Support}\,\mathrm{HM}(p,T) \subseteq \mathrm{Support}\,p$.

(21) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, and p be a polynomial of n, L. Then $\mathrm{Support}\,\mathrm{HM}(p,T) = \emptyset$ or $\mathrm{Support}\,\mathrm{HM}(p,T) = \{\mathrm{HT}(p,T)\}$.

(22) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, and p be a polynomial of n, L. Then $\mathrm{term}\,\mathrm{HM}(p,T) = \mathrm{HT}(p,T)$ and $\mathrm{coefficient}\,\mathrm{HM}(p,T) = \mathrm{HC}(p,T)$.

(23) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, and m be a monomial of n, L. Then $\mathrm{HT}(m,T) = \mathrm{term}\,m$ and $\mathrm{HC}(m,T) = \mathrm{coefficient}\,m$ and $\mathrm{HM}(m,T) = m$.

(24) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, and c be a constant polynomial of n, L. Then $\mathrm{HT}(c,T) = \mathrm{EmptyBag}\,n$ and $\mathrm{HC}(c,T) = c(\mathrm{EmptyBag}\,n)$.

(25) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, and a be an element of L. Then $\mathrm{HT}(a\,\_(n,L),T) = \mathrm{EmptyBag}\,n$ and $\mathrm{HC}(a\,\_(n,L),T) = a$.

(26) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, and p be a polynomial of n, L. Then $\mathrm{HT}(\mathrm{HM}(p,T),T) = \mathrm{HT}(p,T)$.

(27) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, and p be a polynomial of n, L. Then $\mathrm{HC}(\mathrm{HM}(p,T),T) = \mathrm{HC}(p,T)$.

(28) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, and p be a polynomial of n, L. Then $\mathrm{HM}(\mathrm{HM}(p,T),T) = \mathrm{HM}(p,T)$.

(29) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable left zeroed right zeroed unital distributive integral domain-like non trivial double loop structure, and p, q be non-zero polynomials of n, L. Then $\mathrm{HT}(p,T) + \mathrm{HT}(q,T) \in$

Support$(p * q)$.

(30) Let n be an ordinal number, L be an add-associative right complementable right zeroed unital distributive non empty double loop structure, and p, q be polynomials of n, L. Then $\mathrm{Support}(p * q) \subseteq \{s + t; s$ ranges over elements of Bags n, t ranges over elements of Bags $n : s \in \mathrm{Support}\, p \wedge t \in \mathrm{Support}\, q\}$.

(31) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, and p, q be non-zero polynomials of n, L. Then $\mathrm{HT}(p*q, T) = \mathrm{HT}(p, T) + \mathrm{HT}(q, T)$.

(32) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, and p, q be non-zero polynomials of n, L. Then $\mathrm{HC}(p*q, T) = \mathrm{HC}(p, T) \cdot \mathrm{HC}(q, T)$.

(33) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, and p, q be non-zero polynomials of n, L. Then $\mathrm{HM}(p*q, T) = \mathrm{HM}(p, T) * \mathrm{HM}(q, T)$.

(34) Let n be an ordinal number, T be an admissible connected term order of n, L be a right zeroed non empty loop structure, and p, q be polynomials of n, L. Then $\mathrm{HT}(p + q, T) \leqslant_T \max_T(\mathrm{HT}(p, T), \mathrm{HT}(q, T))$.

4. Reductum of a Polynomial

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed non empty loop structure, and let p be a polynomial of n, L. The functor $\mathrm{Red}(p, T)$ yielding a polynomial of n, L is defined by:

(Def. 9) $\mathrm{Red}(p, T) = p - \mathrm{HM}(p, T)$.

The following propositions are true:

(35) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p be a polynomial of n, L. Then $\mathrm{Support}\,\mathrm{Red}(p, T) \subseteq \mathrm{Support}\, p$.

(36) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p be a polynomial of n, L. Then $\mathrm{Support}\,\mathrm{Red}(p, T) = \mathrm{Support}\, p \setminus \{\mathrm{HT}(p, T)\}$.

(37) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop

structure, and p be a polynomial of n, L. Then $\mathrm{Support}(\mathrm{HM}(p,T) + \mathrm{Red}(p,T)) = \mathrm{Support}\, p$.

(38) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p be a polynomial of n, L. Then $\mathrm{HM}(p,T) + \mathrm{Red}(p,T) = p$.

(39) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p be a polynomial of n, L. Then $(\mathrm{Red}(p,T))(\mathrm{HT}(p,T)) = 0_L$.

(40) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, p be a polynomial of n, L, and b be a bag of n. If $b \in \mathrm{Support}\, p$ and $b \neq \mathrm{HT}(p,T)$, then $(\mathrm{Red}(p,T))(b) = p(b)$.

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References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Thomas Becker and Volker Weispfenning. *Gröbner Bases: A Computational Approach to Commutative Algebra*. Springer-Verlag, New York, Berlin, 1993.
[6] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.
[7] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[9] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[10] Gilbert Lee and Piotr Rudnicki. On ordering of bags. *Formalized Mathematics*, 10(**1**):39–46, 2002.
[11] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):3–11, 1991.
[12] Michał Muzalewski and Wojciech Skaba. From loops to abelian multiplicative groups with zero. *Formalized Mathematics*, 1(**5**):833–840, 1990.
[13] Takaya Nishiyama and Yasuho Mizuhara. Binary arithmetics. *Formalized Mathematics*, 4(**1**):83–86, 1993.
[14] Piotr Rudnicki and Andrzej Trybulec. Multivariate polynomials with arbitrary number of variables. *Formalized Mathematics*, 9(**1**):95–110, 2001.
[15] Christoph Schwarzweller. More on multivariate polynomials: Monomials and constant polynomials. *Formalized Mathematics*, 9(**4**):849–855, 2001.
[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[17] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[18] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[19] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.

[20] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[21] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[22] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[23] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[24] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(**1**):85–89, 1990.

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