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Polynomial Reduction

Christoph Schwarzweller
University of Tuebingen

Summary. We continue the formalization of [8] towards Gröbner Bases. In this article we introduce reduction of polynomials and prove its termination, its adequateness for ideal congruence as well as the translation lemma used later to show confluence of reduction.

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The notation and terminology used here are introduced in the following papers: [21], [26], [12], [27], [29], [28], [10], [11], [4], [3], [17], [6], [22], [13], [5], [25], [2], [7], [24], [9], [16], [14], [19], [1], [23], [18], [15], and [20].

1. Preliminaries

Let n be an ordinal number and let R be a non trivial zero structure. One can verify that there exists a monomial of n, R which is non-zero.

Let us observe that there exists a field which is non trivial.

Let us note that every left zeroed add-right-cancelable right distributive left unital commutative associative non empty double loop structure which is field-like is also integral domain-like.

Let n be an ordinal number, let L be an add-associative right complementable left zeroed right zeroed unital distributive integral domain-like non trivial double loop structure, and let p, q be non-zero finite-Support series of n, L. Note that $p * q$ is non-zero.

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2. More on Polynomials and Monomials

The following propositions are true:

(1) Let X be a set, L be an Abelian add-associative right zeroed right complementable non empty loop structure, and p, q be series of X, L. Then $-(p+q) = -p + -q$.

(2) For every set X and for every left zeroed non empty loop structure L and for every series p of X, L holds $0\_(X,L) + p = p$.

(3) Let X be a set, L be an add-associative right zeroed right complementable non empty loop structure, and p be a series of X, L. Then $-p + p = 0\_(X,L)$ and $p + -p = 0\_(X,L)$.

(4) Let n be a set, L be an add-associative right zeroed right complementable non empty loop structure, and p be a series of n, L. Then $p - 0\_(n,L) = p$.

(5) Let n be an ordinal number, L be an add-associative right complementable right zeroed add-left-cancelable left distributive non empty double loop structure, and p be a series of n, L. Then $0\_(n,L) * p = 0\_(n,L)$.

(6) Let n be an ordinal number, L be an Abelian right zeroed add-associative right complementable unital distributive associative commutative non trivial double loop structure, and p, q be polynomials of n, L. Then $-p * q = (-p) * q$ and $-p * q = p * -q$.

(7) Let n be an ordinal number, L be an add-associative right complementable right zeroed distributive non empty double loop structure, p be a polynomial of n, L, m be a monomial of n, L, and b be a bag of n. Then $(m * p)(\mathrm{term}\, m + b) = m(\mathrm{term}\, m) \cdot p(b)$.

(8) Let X be a set, L be a right zeroed add-left-cancelable left distributive non empty double loop structure, and p be a series of X, L. Then $0_L \cdot p = 0\_(X,L)$.

(9) Let X be a set, L be an add-associative right zeroed right complementable distributive non empty double loop structure, p be a series of X, L, and a be an element of L. Then $-a \cdot p = (-a) \cdot p$ and $-a \cdot p = a \cdot -p$.

(10) Let X be a set, L be a left distributive non empty double loop structure, p be a series of X, L, and a, a' be elements of L. Then $a \cdot p + a' \cdot p = (a + a') \cdot p$.

(11) Let X be a set, L be an associative non empty multiplicative loop with zero structure, p be a series of X, L, and a, a' be elements of L. Then $(a \cdot a') \cdot p = a \cdot (a' \cdot p)$.

(12) Let n be an ordinal number, L be an add-associative right zeroed right complementable unital associative commutative distributive non empty double loop structure, p, p' be series of n, L, and a be an element of L. Then $a \cdot (p * p') = p * (a \cdot p')$.

3. Multiplication of Polynomials with Bags

Let n be an ordinal number, let b be a bag of n, let L be a non empty zero structure, and let p be a series of n, L. The functor $b * p$ yielding a series of n, L is defined as follows:

(Def. 1) For every bag b' of n such that $b \mid b'$ holds $(b * p)(b') = p(b' -' b)$ and for every bag b' of n such that $b \nmid b'$ holds $(b * p)(b') = 0_L$.

Let n be an ordinal number, let b be a bag of n, let L be a non empty zero structure, and let p be a finite-Support series of n, L. Note that $b * p$ is finite-Support.

We now state a number of propositions:

(13) Let n be an ordinal number, b, b' be bags of n, L be a non empty zero structure, and p be a series of n, L. Then $(b * p)(b' + b) = p(b')$.

(14) Let n be an ordinal number, L be a non empty zero structure, p be a polynomial of n, L, and b be a bag of n. Then $\mathrm{Support}(b * p) \subseteq \{b + b'; b'$ ranges over elements of $\mathrm{Bags}\, n : b' \in \mathrm{Support}\, p\}$.

(15) Let n be an ordinal number, T be a connected admissible term order of n, L be a non trivial zero structure, p be a non-zero polynomial of n, L, and b be a bag of n. Then $\mathrm{HT}(b * p, T) = b + \mathrm{HT}(p, T)$.

(16) Let n be an ordinal number, T be a connected admissible term order of n, L be a non empty zero structure, p be a polynomial of n, L, and b, b' be bags of n. If $b' \in \mathrm{Support}(b * p)$, then $b' \leqslant_T b + \mathrm{HT}(p, T)$.

(17) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, and p be a series of n, L. Then $\mathrm{EmptyBag}\, n * p = p$.

(18) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, p be a series of n, L, and b_1, b_2 be bags of n. Then $(b_1 + b_2) * p = b_1 * (b_2 * p)$.

(19) Let n be an ordinal number, L be an add-associative right zeroed right complementable distributive non trivial double loop structure, p be a polynomial of n, L, and a be an element of L. Then $\mathrm{Support}(a \cdot p) \subseteq \mathrm{Support}\, p$.

(20) Let n be an ordinal number, L be an integral domain-like non trivial double loop structure, p be a polynomial of n, L, and a be a non-zero element of L. Then $\mathrm{Support}\, p \subseteq \mathrm{Support}(a \cdot p)$.

(21) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable distributive integral domain-like non trivial double loop structure, p be a polynomial of n, L, and a be a non-zero element of L. Then $\mathrm{HT}(a \cdot p, T) = \mathrm{HT}(p, T)$.

(22) Let n be an ordinal number, L be an add-associative right complementable right zeroed distributive non trivial double loop structure, p

be a series of n, L, b be a bag of n, and a be an element of L. Then $a \cdot (b * p) = \mathrm{Monom}(a, b) * p$.

(23) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, p be a non-zero polynomial of n, L, q be a polynomial of n, L, and m be a non-zero monomial of n, L. If $\mathrm{HT}(p, T) \in \mathrm{Support}\, q$, then $\mathrm{HT}(m * p, T) \in \mathrm{Support}(m * q)$.

4. Orders on Polynomials

Let n be an ordinal number and let T be a connected term order of n. Observe that $\langle \mathrm{Bags}\, n, T \rangle$ is connected.

Let n be a natural number and let T be an admissible term order of n. Note that $\langle \mathrm{Bags}\, n, T \rangle$ is well founded.

Let n be an ordinal number, let T be a connected term order of n, let L be a non empty zero structure, and let p, q be polynomials of n, L. The predicate $p \leqslant_T q$ is defined as follows:

(Def. 2) $\langle \mathrm{Support}\, p, \mathrm{Support}\, q \rangle \in \mathrm{FinOrd}\langle \mathrm{Bags}\, n, T \rangle$.

Let n be an ordinal number, let T be a connected term order of n, let L be a non empty zero structure, and let p, q be polynomials of n, L. The predicate $p <_T q$ is defined as follows:

(Def. 3) $p \leqslant_T q$ and $\mathrm{Support}\, p \neq \mathrm{Support}\, q$.

Let n be an ordinal number, let T be a connected term order of n, let L be a non empty zero structure, and let p be a polynomial of n, L. The functor $\mathrm{Support}(p, T)$ yielding an element of $\mathrm{Fin}\,(\text{the carrier of } \langle \mathrm{Bags}\, n, T \rangle)$ is defined by:

(Def. 4) $\mathrm{Support}(p, T) = \mathrm{Support}\, p$.

Next we state a number of propositions:

(24) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, and p be a non-zero polynomial of n, L. Then $\mathrm{PosetMax}\, \mathrm{Support}(p, T) = \mathrm{HT}(p, T)$.

(25) Let n be an ordinal number, T be a connected term order of n, L be a non empty loop structure, and p be a polynomial of n, L. Then $p \leqslant_T p$.

(26) Let n be an ordinal number, T be a connected term order of n, L be a non empty loop structure, and p, q be polynomials of n, L. Then $p \leqslant_T q$ and $q \leqslant_T p$ if and only if $\mathrm{Support}\, p = \mathrm{Support}\, q$.

(27) Let n be an ordinal number, T be a connected term order of n, L be a non empty loop structure, and p, q, r be polynomials of n, L. If $p \leqslant_T q$ and $q \leqslant_T r$, then $p \leqslant_T r$.

(28) Let n be an ordinal number, T be a connected term order of n, L be a non empty loop structure, and p, q be polynomials of n, L. Then $p \leqslant_T q$ or $q \leqslant_T p$.

(29) Let n be an ordinal number, T be a connected term order of n, L be a non empty loop structure, and p, q be polynomials of n, L. Then $p \leqslant_T q$ if and only if $q \not<_T p$.

(30) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, and p be a polynomial of n, L. Then $0\_(n, L) \leqslant_T p$.

(31) Let n be a natural number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed unital distributive non trivial double loop structure, and P be a non empty subset of Polynom-Ring(n, L). Then there exists a polynomial p of n, L such that $p \in P$ and for every polynomial q of n, L such that $q \in P$ holds $p \leqslant_T q$.

(32) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p, q be polynomials of n, L. Then $p <_T q$ if and only if one of the following conditions is satisfied:

  (i) $p = 0\_(n, L)$ and $q \neq 0\_(n, L)$, or
  (ii) $\mathrm{HT}(p, T) <_T \mathrm{HT}(q, T)$, or
  (iii) $\mathrm{HT}(p, T) = \mathrm{HT}(q, T)$ and $\mathrm{Red}(p, T) <_T \mathrm{Red}(q, T)$.

(33) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p be a non-zero polynomial of n, L. Then $\mathrm{Red}(p, T) <_T \mathrm{HM}(p, T)$.

(34) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p be a polynomial of n, L. Then $\mathrm{HM}(p, T) \leqslant_T p$.

(35) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p be a non-zero polynomial of n, L. Then $\mathrm{Red}(p, T) <_T p$.

5. Polynomial Reduction

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, let f, p, g be polynomials of n, L, and let b be a bag of n. We say that f reduces to g, p, b, T if and only if:

(Def. 5) $f \neq 0\_(n, L)$ and $p \neq 0\_(n, L)$ and $b \in \mathrm{Support}\, f$ and there exists a bag s of n such that $s + \mathrm{HT}(p, T) = b$ and $g = f - \frac{f(b)}{\mathrm{HC}(p,T)} \cdot (s * p)$.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let f, p, g be polynomials of n, L. We say that f reduces to g, p, T if and only if:

(Def. 6) There exists a bag b of n such that f reduces to g, p, b, T.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, let f, g be polynomials of n, L, and let P be a subset of Polynom-Ring(n, L). We say that f reduces to g, P, T if and only if:

(Def. 7) There exists a polynomial p of n, L such that $p \in P$ and f reduces to g, p, T.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let f, p be polynomials of n, L. We say that f is reducible wrt p, T if and only if:

(Def. 8) There exists a polynomial g of n, L such that f reduces to g, p, T.

We introduce f is irreducible wrt p, T and f is in normal form wrt p, T as antonyms of f is reducible wrt p, T.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, let f be a polynomial of n, L, and let P be a subset of Polynom-Ring(n, L). We say that f is reducible wrt P, T if and only if:

(Def. 9) There exists a polynomial g of n, L such that f reduces to g, P, T.

We introduce f is irreducible wrt P, T and f is in normal form wrt P, T as antonyms of f is reducible wrt P, T.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let f, p, g be polynomials of n, L. We say that f top reduces to g, p, T if and only if:

(Def. 10) f reduces to g, p, $\mathrm{HT}(f, T)$, T.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let f, p be polynomials of n, L. We say that f is top reducible wrt p, T if and only if:

(Def. 11) There exists a polynomial g of n, L such that f top reduces to g, p, T.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, let f be a polynomial of n, L, and let P be a subset of Polynom-Ring(n, L). We say that f is top reducible wrt P, T if and only if:

(Def. 12) There exists a polynomial p of n, L such that $p \in P$ and f is top reducible wrt p, T.

Next we state several propositions:

(36) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, f be a polynomial of n, L, and p be a non-zero polynomial of n, L. Then f is reducible wrt p, T if and only if there exists a bag b of n such that $b \in \mathrm{Support}\, f$ and $\mathrm{HT}(p, T) \mid b$.

(37) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and p be a polynomial of n, L. Then $0\_(n, L)$ is irreducible wrt p, T.

(38) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, f, p be polynomials of n, L, and m be a non-zero monomial of n, L. If f reduces to $f - m * p$, p, T, then $\mathrm{HT}(m * p, T) \in \mathrm{Support}\, f$.

(39) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, f, p, g be polynomials of n, L, and b be a bag of n. If f reduces to g, p, b, T, then $b \notin \mathrm{Support}\, g$.

(40) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, f, p, g be polynomials of n, L, and b, b' be bags of n. Suppose $b <_T b'$. If f reduces to g, p, b, T, then $b' \in \mathrm{Support}\, g$ iff $b' \in \mathrm{Support}\, f$.

(41) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, f, p, g be polynomials of n, L, and b, b' be bags of n. If $b <_T b'$, then if f reduces to g, p, b, T, then $f(b') = g(b')$.

(42) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and f, p, g be polynomials of n, L. Suppose f

reduces to g, p, T. Let b be a bag of n. If $b \in \operatorname{Support} g$, then $b \leqslant_T \mathrm{HT}(f, T)$.

(43) Let n be an ordinal number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and f, p, g be polynomials of n, L. If f reduces to g, p, T, then $g <_T f$.

6. Polynomial Reduction Relation

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let P be a subset of Polynom-Ring(n, L). The functor PolyRedRel(P, T) yields a relation between (the carrier of Polynom-Ring(n, L)) $\setminus \{0\_(n, L)\}$ and the carrier of Polynom-Ring(n, L) and is defined by:

(Def. 13) For all polynomials p, q of n, L holds $\langle p, q \rangle \in$ PolyRedRel(P, T) iff p reduces to q, P, T.

Next we state the proposition

(44) Let n be an ordinal number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, f, g be polynomials of n, L, and P be a subset of Polynom-Ring(n, L). If PolyRedRel(P, T) reduces f to g, then $g \leqslant_T f$ but $g = 0\_(n, L)$ or $\mathrm{HT}(g, T) \leqslant_T \mathrm{HT}(f, T)$.

Let n be a natural number, let T be a connected admissible term order of n, let L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and let P be a subset of Polynom-Ring(n, L). One can verify that PolyRedRel(P, T) is strongly-normalizing.

One can prove the following propositions:

(45) Let n be a natural number, T be an admissible connected term order of n, L be an add-associative right complementable left zeroed right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure, P be a subset of Polynom-Ring(n, L), and f, h be polynomials of n, L. If $f \in P$, then PolyRedRel(P, T) reduces $h * f$ to $0\_(n, L)$.

(46) Let n be an ordinal number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated

non empty double loop structure, P be a subset of Polynom-Ring(n, L), f, g be polynomials of n, L, and m be a non-zero monomial of n, L. If f reduces to g, P, T, then $m * f$ reduces to $m * g$, P, T.

(47) Let n be an ordinal number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, P be a subset of Polynom-Ring(n, L), f, g be polynomials of n, L, and m be a monomial of n, L. If PolyRedRel(P, T) reduces f to g, then PolyRedRel(P, T) reduces $m * f$ to $m * g$.

(48) Let n be an ordinal number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, P be a subset of Polynom-Ring(n, L), f be a polynomial of n, L, and m be a monomial of n, L. If PolyRedRel(P, T) reduces f to $0\_(n, L)$, then PolyRedRel(P, T) reduces $m * f$ to $0\_(n, L)$.

(49) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure, P be a subset of Polynom-Ring(n, L), and f, g, h, h_1 be polynomials of n, L. Suppose $f - g = h$ and PolyRedRel(P, T) reduces h to h_1. Then there exist polynomials f_1, g_1 of n, L such that $f_1 - g_1 = h_1$ and PolyRedRel(P, T) reduces f to f_1 and PolyRedRel(P, T) reduces g to g_1.

(50) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure, P be a subset of Polynom-Ring(n, L), and f, g be polynomials of n, L. Suppose PolyRedRel(P, T) reduces $f - g$ to $0\_(n, L)$. Then f and g are convergent w.r.t. PolyRedRel(P, T).

(51) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure, P be a subset of Polynom-Ring(n, L), and f, g be polynomials of n, L. Suppose PolyRedRel(P, T) reduces $f - g$ to $0\_(n, L)$. Then f and g are convertible w.r.t. PolyRedRel(P, T).

Let R be a non empty loop structure, let I be a subset of R, and let a, b be elements of R. The predicate $a \equiv b (\mathrm{mod}\, I)$ is defined as follows:

(Def. 14) $a - b \in I$.

One can prove the following propositions:

(52) Let R be an add-associative left zeroed right zeroed right complementable right distributive non empty double loop structure, I be a right ideal

non empty subset of R, and a be an element of R. Then $a \equiv a (\mathrm{mod}\, I)$.

(53) Let R be an add-associative right zeroed right complementable right unital right distributive non empty double loop structure, I be a right ideal non empty subset of R, and a, b be elements of R. If $a \equiv b (\mathrm{mod}\, I)$, then $b \equiv a (\mathrm{mod}\, I)$.

(54) Let R be an add-associative right zeroed right complementable non empty loop structure, I be an add closed non empty subset of R, and a, b, c be elements of R. If $a \equiv b (\mathrm{mod}\, I)$ and $b \equiv c (\mathrm{mod}\, I)$, then $a \equiv c (\mathrm{mod}\, I)$.

(55) Let R be an Abelian add-associative right zeroed right complementable unital distributive associative non trivial double loop structure, I be an add closed non empty subset of R, and a, b, c, d be elements of R. If $a \equiv b (\mathrm{mod}\, I)$ and $c \equiv d (\mathrm{mod}\, I)$, then $a + c \equiv b + d (\mathrm{mod}\, I)$.

(56) Let R be an add-associative right zeroed right complementable commutative distributive non empty double loop structure, I be an add closed right ideal non empty subset of R, and a, b, c, d be elements of R. If $a \equiv b (\mathrm{mod}\, I)$ and $c \equiv d (\mathrm{mod}\, I)$, then $a \cdot c \equiv b \cdot d (\mathrm{mod}\, I)$.

(57) Let n be an ordinal number, T be a connected term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, P be a subset of Polynom-Ring(n, L), and f, g be elements of Polynom-Ring(n, L). If f and g are convertible w.r.t. PolyRedRel(P, T), then $f \equiv g (\mathrm{mod}\, P\text{–ideal})$.

(58) Let n be a natural number, T be an admissible connected term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, P be a non empty subset of Polynom-Ring(n, L), and f, g be elements of Polynom-Ring(n, L). If $f \equiv g (\mathrm{mod}\, P\text{–ideal})$, then f and g are convertible w.r.t. PolyRedRel(P, T).

(59) Let n be an ordinal number, T be a connected term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, P be a subset of Polynom-Ring(n, L), and f, g be polynomials of n, L. If PolyRedRel(P, T) reduces f to g, then $f - g \in P$–ideal.

(60) Let n be an ordinal number, T be a connected term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, P be a subset of Polynom-Ring(n, L), and f be a polynomial of n, L. If PolyRedRel(P, T) reduces f to $0\_(n, L)$, then $f \in P$–ideal.

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