FORMALIZED MATHEMATICS
Volume 11, Number 1, 2003
University of Białystok

Bilinear Functionals in Vector Spaces[1]

Jarosław Kotowicz
University of Białystok

Summary. The main goal of the article is the presentation of the theory of bilinear functionals in vector spaces. It introduces standard operations on bilinear functionals and proves their classical properties. It is shown that quotient functionals are non-degenerate on the left and the right. In the case of symmetric and alternating bilinear functionals it is shown that the left and right kernels are equal.

MML Identifier: `BILINEAR`.

The papers [13], [6], [17], [12], [4], [18], [11], [2], [16], [3], [9], [19], [5], [7], [1], [15], [14], [10], and [8] provide the notation and terminology for this paper.

1. Two Form on Vector Spaces and Operations on Them

Let K be a 1-sorted structure and let V, W be vector space structures over K.

(Def. 1) A function from $[\!:$ the carrier of V, the carrier of $W :\!]$ into the carrier of K is said to be a form of V, W.

Let K be a non empty zero structure and let V, W be vector space structures over K. The functor $\mathrm{NulForm}(V, W)$ yielding a form of V, W is defined by:

(Def. 2) $\mathrm{NulForm}(V, W) = [\!:$ the carrier of V, the carrier of $W :\!] \longmapsto 0_K$.

Let K be a non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be forms of V, W. The functor $f + g$ yields a form of V, W and is defined as follows:

(Def. 3) For every vector v of V and for every vector w of W holds $(f + g)(\langle v, w\rangle) = f(\langle v, w\rangle) + g(\langle v, w\rangle)$.

[1]This work has been partially supported by TRIAL-SOLUTION grant IST-2001-35447 and SPUB-M grant of KBN.

ISSN 1426–2630

Let K be a non empty groupoid, let V, W be non empty vector space structures over K, let f be a form of V, W, and let a be an element of the carrier of K. The functor $a \cdot f$ yields a form of V, W and is defined by:

(Def. 4) For every vector v of V and for every vector w of W holds $(a \cdot f)(\langle v, w\rangle) = a \cdot f(\langle v, w\rangle)$.

Let K be a non empty loop structure, let V, W be non empty vector space structures over K, and let f be a form of V, W. The functor $-f$ yielding a form of V, W is defined as follows:

(Def. 5) For every vector v of V and for every vector w of W holds $(-f)(\langle v, w\rangle) = -f(\langle v, w\rangle)$.

Let K be an add-associative right zeroed right complementable left distributive left unital non empty double loop structure, let V, W be non empty vector space structures over K, and let f be a form of V, W. Then $-f$ is a form of V, W and it can be characterized by the condition:

(Def. 6) $-f = (-\mathbf{1}_K) \cdot f$.

Let K be a non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be forms of V, W. The functor $f - g$ yields a form of V, W and is defined by:

(Def. 7) $f - g = f + -g$.

Let K be a non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be forms of V, W. Then $f - g$ is a form of V, W and it can be characterized by the condition:

(Def. 8) For every vector v of V and for every vector w of W holds $(f - g)(\langle v, w\rangle) = f(\langle v, w\rangle) - g(\langle v, w\rangle)$.

Let K be an Abelian non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be forms of V, W. Let us notice that the functor $f + g$ is commutative.

Next we state several propositions:

(1) Let K be a non empty zero structure, V, W be non empty vector space structures over K, v be a vector of V, and w be a vector of W. Then $(\mathrm{NulForm}(V, W))(\langle v, w\rangle) = 0_K$.

(2) Let K be a right zeroed non empty loop structure, V, W be non empty vector space structures over K, and f be a form of V, W. Then $f + \mathrm{NulForm}(V, W) = f$.

(3) Let K be an add-associative non empty loop structure, V, W be non empty vector space structures over K, and f, g, h be forms of V, W. Then $(f + g) + h = f + (g + h)$.

(4) Let K be an add-associative right zeroed right complementable non empty loop structure, V, W be non empty vector space structures over K, and f be a form of V, W. Then $f - f = \mathrm{NulForm}(V, W)$.

(5) Let K be a right distributive non empty double loop structure, V, W be non empty vector space structures over K, a be an element of the carrier of K, and f, g be forms of V, W. Then $a \cdot (f+g) = a \cdot f + a \cdot g$.

(6) Let K be a left distributive non empty double loop structure, V, W be non empty vector space structures over K, a, b be elements of the carrier of K, and f be a form of V, W. Then $(a+b) \cdot f = a \cdot f + b \cdot f$.

(7) Let K be an associative non empty double loop structure, V, W be non empty vector space structures over K, a, b be elements of the carrier of K, and f be a form of V, W. Then $(a \cdot b) \cdot f = a \cdot (b \cdot f)$.

(8) Let K be a left unital non empty double loop structure, V, W be non empty vector space structures over K, and f be a form of V, W. Then $\mathbf{1}_K \cdot f = f$.

2. Functional Generated by Two Form when the One of Arguments is Fixed

Let K be a non empty 1-sorted structure, let V, W be non empty vector space structures over K, let f be a form of V, W, and let v be a vector of V. The functor $f(v,\cdot)$ yielding a functional in W is defined as follows:

(Def. 9) $f(v,\cdot) = (\mathrm{curry}\, f)(v)$.

Let K be a non empty 1-sorted structure, let V, W be non empty vector space structures over K, let f be a form of V, W, and let w be a vector of W. The functor $f(\cdot,w)$ yields a functional in V and is defined by:

(Def. 10) $f(\cdot,w) = (\mathrm{curry}'\, f)(w)$.

The following propositions are true:

(9) Let K be a non empty 1-sorted structure, V, W be non empty vector space structures over K, f be a form of V, W, and v be a vector of V. Then $\mathrm{dom}\, f(v,\cdot) =$ the carrier of W and $\mathrm{rng}\, f(v,\cdot) \subseteq$ the carrier of K and for every vector w of W holds $(f(v,\cdot))(w) = f(\langle v,\ w\rangle)$.

(10) Let K be a non empty 1-sorted structure, V, W be non empty vector space structures over K, f be a form of V, W, and w be a vector of W. Then $\mathrm{dom}\, f(\cdot,w) =$ the carrier of V and $\mathrm{rng}\, f(\cdot,w) \subseteq$ the carrier of K and for every vector v of V holds $(f(\cdot,w))(v) = f(\langle v,\ w\rangle)$.

(11) Let K be a non empty zero structure, V, W be non empty vector space structures over K, f be a form of V, W, and v be a vector of V. Then $\mathrm{NulForm}(V,W)(v,\cdot) = 0\mathrm{Functional}\, W$.

(12) Let K be a non empty zero structure, V, W be non empty vector space structures over K, f be a form of V, W, and w be a vector of W. Then $\mathrm{NulForm}(V,W)(\cdot,w) = 0\mathrm{Functional}\, V$.

(13) Let K be a non empty loop structure, V, W be non empty vector space structures over K, f, g be forms of V, W, and w be a vector of W. Then $(f+g)(\cdot,w) = f(\cdot,w) + g(\cdot,w)$.

(14) Let K be a non empty loop structure, V, W be non empty vector space structures over K, f, g be forms of V, W, and v be a vector of V. Then $(f+g)(v,\cdot) = f(v,\cdot) + g(v,\cdot)$.

(15) Let K be a non empty double loop structure, V, W be non empty vector space structures over K, f be a form of V, W, a be an element of the carrier of K, and w be a vector of W. Then $(a \cdot f)(\cdot,w) = a \cdot f(\cdot,w)$.

(16) Let K be a non empty double loop structure, V, W be non empty vector space structures over K, f be a form of V, W, a be an element of the carrier of K, and v be a vector of V. Then $(a \cdot f)(v,\cdot) = a \cdot f(v,\cdot)$.

(17) Let K be a non empty loop structure, V, W be non empty vector space structures over K, f be a form of V, W, and w be a vector of W. Then $(-f)(\cdot,w) = -f(\cdot,w)$.

(18) Let K be a non empty loop structure, V, W be non empty vector space structures over K, f be a form of V, W, and v be a vector of V. Then $(-f)(v,\cdot) = -f(v,\cdot)$.

(19) Let K be a non empty loop structure, V, W be non empty vector space structures over K, f, g be forms of V, W, and w be a vector of W. Then $(f-g)(\cdot,w) = f(\cdot,w) - g(\cdot,w)$.

(20) Let K be a non empty loop structure, V, W be non empty vector space structures over K, f, g be forms of V, W, and v be a vector of V. Then $(f-g)(v,\cdot) = f(v,\cdot) - g(v,\cdot)$.

3. Two Form Generated by Functionals

Let K be a non empty groupoid, let V, W be non empty vector space structures over K, let f be a functional in V, and let g be a functional in W. The functor $f \otimes g$ yields a form of V, W and is defined as follows:

(Def. 11) For every vector v of V and for every vector w of W holds $f \otimes g(\langle v, w\rangle) = f(v) \cdot g(w)$.

One can prove the following propositions:

(21) Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, V, W be non empty vector space structures over K, f be a functional in V, v be a vector of V, and w be a vector of W. Then $f \otimes (0\mathrm{Functional}\, W)(\langle v, w\rangle) = 0_K$.

(22) Let K be an add-associative right zeroed right complementable left distributive non empty double loop structure, V, W be non empty vector

space structures over K, g be a functional in W, v be a vector of V, and w be a vector of W. Then $(\mathrm{0Functional}\,V) \otimes g(\langle v, w\rangle) = 0_K$.

(23) Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, V, W be non empty vector space structures over K, and f be a functional in V. Then $f \otimes (\mathrm{0Functional}\,W) = \mathrm{NulForm}(V, W)$.

(24) Let K be an add-associative right zeroed right complementable left distributive non empty double loop structure, V, W be non empty vector space structures over K, and g be a functional in W. Then $(\mathrm{0Functional}\,V) \otimes g = \mathrm{NulForm}(V, W)$.

(25) Let K be a non empty groupoid, V, W be non empty vector space structures over K, f be a functional in V, g be a functional in W, and v be a vector of V. Then $(f \otimes g)(v, \cdot) = f(v) \cdot g$.

(26) Let K be a commutative non empty groupoid, V, W be non empty vector space structures over K, f be a functional in V, g be a functional in W, and w be a vector of W. Then $(f \otimes g)(\cdot, w) = g(w) \cdot f$.

4. Bilinear Forms and their Properties

Let K be a non empty loop structure, let V, W be non empty vector space structures over K, and let f be a form of V, W. We say that f is additive wrt. second argument if and only if:

(Def. 12) For every vector v of V holds $f(v, \cdot)$ is additive.

We say that f is additive wrt. first argument if and only if:

(Def. 13) For every vector w of W holds $f(\cdot, w)$ is additive.

Let K be a non empty groupoid, let V, W be non empty vector space structures over K, and let f be a form of V, W. We say that f is homogeneous wrt. second argument if and only if:

(Def. 14) For every vector v of V holds $f(v, \cdot)$ is homogeneous.

We say that f is homogeneous wrt. first argument if and only if:

(Def. 15) For every vector w of W holds $f(\cdot, w)$ is homogeneous.

Let K be a right zeroed non empty loop structure and let V, W be non empty vector space structures over K. Note that $\mathrm{NulForm}(V, W)$ is additive wrt. second argument and $\mathrm{NulForm}(V, W)$ is additive wrt. first argument.

Let K be a right zeroed non empty loop structure and let V, W be non empty vector space structures over K. Note that there exists a form of V, W which is additive wrt. second argument and additive wrt. first argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure and let V, W be non empty vector space structures over K. Observe that $\mathrm{NulForm}(V, W)$ is homogeneous wrt. second argument and $\mathrm{NulForm}(V, W)$ is homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure and let V, W be non empty vector space structures over K. One can verify that there exists a form of V, W which is additive wrt. second argument, homogeneous wrt. second argument, additive wrt. first argument, and homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure and let V, W be non empty vector space structures over K. A bilinear form of V, W is an additive wrt. first argument homogeneous wrt. first argument additive wrt. second argument homogeneous wrt. second argument form of V, W.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be an additive wrt. second argument form of V, W, and let v be a vector of V. Note that $f(v, \cdot)$ is additive.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be an additive wrt. first argument form of V, W, and let w be a vector of W. One can check that $f(\cdot, w)$ is additive.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be a homogeneous wrt. second argument form of V, W, and let v be a vector of V. Note that $f(v, \cdot)$ is homogeneous.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be a homogeneous wrt. first argument form of V, W, and let w be a vector of W. One can verify that $f(\cdot, w)$ is homogeneous.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be a functional in V, and let g be an additive functional in W. One can check that $f \otimes g$ is additive wrt. second argument.

Let K be an add-associative right zeroed right complementable commutative right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be an additive functional in V, and let g be a functional in W. Note that $f \otimes g$ is additive wrt. first argument.

Let K be an add-associative right zeroed right complementable commutative associative right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be a functional in V, and let g be a homogeneous functional in W. Note that $f \otimes g$ is homogeneous wrt. second argument.

Let K be an add-associative right zeroed right complementable commutative associative right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be a homogeneous functional in V, and let g be a functional in W. Note that $f \otimes g$ is homogeneous wrt. first argument.

Let K be a non degenerated non empty double loop structure, let V be a non trivial non empty vector space structure over K, let W be a non empty vector space structure over K, let f be a functional in V, and let g be a functional in W. One can verify that $f \otimes g$ is non trivial.

Let K be a non degenerated non empty double loop structure, let V be a non empty vector space structure over K, let W be a non trivial non empty vector space structure over K, let f be a functional in V, and let g be a functional in W. One can verify that $f \otimes g$ is non trivial.

Let K be a field, let V, W be non trivial vector spaces over K, let f be a non constant 0-preserving functional in V, and let g be a non constant 0-preserving functional in W. Observe that $f \otimes g$ is non constant.

Let K be a field and let V, W be non trivial vector spaces over K. Observe that there exists a form of V, W which is non trivial, non constant, additive wrt. second argument, homogeneous wrt. second argument, additive wrt. first argument, and homogeneous wrt. first argument.

Let K be an Abelian add-associative right zeroed non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be additive wrt. first argument forms of V, W. Observe that $f + g$ is additive wrt. first argument.

Let K be an Abelian add-associative right zeroed non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be additive wrt. second argument forms of V, W. Observe that $f+g$ is additive wrt. second argument.

Let K be a right distributive right zeroed non empty double loop structure, let V, W be non empty vector space structures over K, let f be an additive wrt. first argument form of V, W, and let a be an element of the carrier of K. Observe that $a \cdot f$ is additive wrt. first argument.

Let K be a right distributive right zeroed non empty double loop structure, let V, W be non empty vector space structures over K, let f be an additive wrt. second argument form of V, W, and let a be an element of the carrier of K. Observe that $a \cdot f$ is additive wrt. second argument.

Let K be an Abelian add-associative right zeroed right complementable non empty loop structure, let V, W be non empty vector space structures over K, and let f be an additive wrt. first argument form of V, W. One can verify that $-f$ is additive wrt. first argument.

Let K be an Abelian add-associative right zeroed right complementable non empty loop structure, let V, W be non empty vector space structures over K, and let f be an additive wrt. second argument form of V, W. One can check that $-f$ is additive wrt. second argument.

Let K be an Abelian add-associative right zeroed right complementable non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be additive wrt. first argument forms of V, W. Observe that $f-g$ is additive wrt. first argument.

Let K be an Abelian add-associative right zeroed right complementable non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be additive wrt. second argument forms of V, W. Note that $f-g$ is additive wrt. second argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, and let f, g be homogeneous wrt. first argument forms of V, W. One can verify that $f+g$ is homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, and let f, g be homogeneous wrt. second argument forms of V, W. Note that $f+g$ is homogeneous wrt. second argument.

Let K be an add-associative right zeroed right complementable associative commutative right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be a homogeneous wrt. first argument form of V, W, and let a be an element of the carrier of K. One can check that $a \cdot f$ is homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable associative commutative right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be a homogeneous wrt. second argument form of V, W, and let a be an element of the carrier of K. One can check that $a \cdot f$ is homogeneous wrt. second argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, and let f be a homogeneous wrt. first argument form of V, W. One can verify that $-f$ is homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, and let f be a homogeneous wrt. second argument form of V, W. Note that $-f$ is homogeneous wrt. second argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, and let f, g be homogeneous wrt. first argument forms of V, W. One can check that $f-g$ is homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, and let f, g be homogeneous wrt. second argument forms of

V, W. One can check that $f - g$ is homogeneous wrt. second argument.

We now state a number of propositions:

(27) Let K be a non empty loop structure, V, W be non empty vector space structures over K, v, u be vectors of V, w be a vector of W, and f be a form of V, W. If f is additive wrt. first argument, then $f(\langle v + u, w\rangle) = f(\langle v, w\rangle) + f(\langle u, w\rangle)$.

(28) Let K be a non empty loop structure, V, W be non empty vector space structures over K, v be a vector of V, u, w be vectors of W, and f be a form of V, W. If f is additive wrt. second argument, then $f(\langle v, u+w\rangle) = f(\langle v, u\rangle) + f(\langle v, w\rangle)$.

(29) Let K be a right zeroed non empty loop structure, V, W be non empty vector space structures over K, v, u be vectors of V, w, t be vectors of W, and f be an additive wrt. first argument additive wrt. second argument form of V, W. Then $f(\langle v + u, w + t\rangle) = f(\langle v, w\rangle) + f(\langle v, t\rangle) + (f(\langle u, w\rangle) + f(\langle u, t\rangle))$.

(30) Let K be an add-associative right zeroed right complementable non empty double loop structure, V, W be right zeroed non empty vector space structures over K, f be an additive wrt. second argument form of V, W, and v be a vector of V. Then $f(\langle v, 0_W\rangle) = 0_K$.

(31) Let K be an add-associative right zeroed right complementable non empty double loop structure, V, W be right zeroed non empty vector space structures over K, f be an additive wrt. first argument form of V, W, and w be a vector of W. Then $f(\langle 0_V, w\rangle) = 0_K$.

(32) Let K be a non empty groupoid, V, W be non empty vector space structures over K, v be a vector of V, w be a vector of W, a be an element of the carrier of K, and f be a form of V, W. If f is homogeneous wrt. first argument, then $f(\langle a \cdot v, w\rangle) = a \cdot f(\langle v, w\rangle)$.

(33) Let K be a non empty groupoid, V, W be non empty vector space structures over K, v be a vector of V, w be a vector of W, a be an element of the carrier of K, and f be a form of V, W. If f is homogeneous wrt. second argument, then $f(\langle v, a \cdot w\rangle) = a \cdot f(\langle v, w\rangle)$.

(34) Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, V, W be add-associative right zeroed right complementable vector space-like non empty vector space structures over K, f be a homogeneous wrt. first argument form of V, W, and w be a vector of W. Then $f(\langle 0_V, w\rangle) = 0_K$.

(35) Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, V, W be add-associative right zeroed right complementable vector space-like non empty vector space structures over K, f be a homogeneous wrt. second

argument form of V, W, and v be a vector of V. Then $f(\langle v, 0_W \rangle) = 0_K$.

(36) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, v, u be vectors of V, w be a vector of W, and f be an additive wrt. first argument homogeneous wrt. first argument form of V, W. Then $f(\langle v - u, w \rangle) = f(\langle v, w \rangle) - f(\langle u, w \rangle)$.

(37) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, v be a vector of V, w, t be vectors of W, and f be an additive wrt. second argument homogeneous wrt. second argument form of V, W. Then $f(\langle v, w - t \rangle) = f(\langle v, w \rangle) - f(\langle v, t \rangle)$.

(38) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, v, u be vectors of V, w, t be vectors of W, and f be a bilinear form of V, W. Then $f(\langle v - u, w - t \rangle) = f(\langle v, w \rangle) - f(\langle v, t \rangle) - (f(\langle u, w \rangle) - f(\langle u, t \rangle))$.

(39) Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, V, W be add-associative right zeroed right complementable vector space-like non empty vector space structures over K, v, u be vectors of V, w, t be vectors of W, a, b be elements of the carrier of K, and f be a bilinear form of V, W. Then $f(\langle v + a \cdot u, w + b \cdot t \rangle) = f(\langle v, w \rangle) + b \cdot f(\langle v, t \rangle) + (a \cdot f(\langle u, w \rangle) + a \cdot (b \cdot f(\langle u, t \rangle)))$.

(40) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, v, u be vectors of V, w, t be vectors of W, a, b be elements of the carrier of K, and f be a bilinear form of V, W. Then $f(\langle v - a \cdot u, w - b \cdot t \rangle) = f(\langle v, w \rangle) - b \cdot f(\langle v, t \rangle) - (a \cdot f(\langle u, w \rangle) - a \cdot (b \cdot f(\langle u, t \rangle)))$.

(41) Let K be an add-associative right zeroed right complementable non empty double loop structure, V, W be right zeroed non empty vector space structures over K, and f be a form of V, W. Suppose f is additive wrt. second argument and additive wrt. first argument. Then f is constant if and only if for every vector v of V and for every vector w of W holds $f(\langle v, w \rangle) = 0_K$.

5. Left and Right Kernel of Form. Kernel of "Diagonal"

Let K be a zero structure, let V, W be non empty vector space structures over K, and let f be a form of V, W. The functor leftker f yields a subset of the carrier of V and is defined as follows:

(Def. 16) $\operatorname{leftker} f = \{v; v$ ranges over vectors of V: $\bigwedge_{w\,:\,\text{vector of}\ W}\ f(\langle v, w\rangle) = 0_K\}$.

Let K be a zero structure, let V, W be non empty vector space structures over K, and let f be a form of V, W. The functor $\operatorname{rightker} f$ yielding a subset of the carrier of W is defined by:

(Def. 17) $\operatorname{rightker} f = \{w; w$ ranges over vectors of W: $\bigwedge_{v\,:\,\text{vector of}\ V}\ f(\langle v, w\rangle) = 0_K\}$.

Let K be a zero structure, let V be a non empty vector space structure over K, and let f be a form of V, V. The functor $\operatorname{diagker} f$ yielding a subset of the carrier of V is defined by:

(Def. 18) $\operatorname{diagker} f = \{v; v$ ranges over vectors of V: $f(\langle v, v\rangle) = 0_K\}$.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V be a right zeroed non empty vector space structure over K, let W be a non empty vector space structure over K, and let f be an additive wrt. first argument form of V, W. Note that $\operatorname{leftker} f$ is non empty.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K, let W be a non empty vector space structure over K, and let f be a homogeneous wrt. first argument form of V, W. Observe that $\operatorname{leftker} f$ is non empty.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V be a non empty vector space structure over K, let W be a right zeroed non empty vector space structure over K, and let f be an additive wrt. second argument form of V, W. One can verify that $\operatorname{rightker} f$ is non empty.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, let V be a non empty vector space structure over K, let W be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K, and let f be a homogeneous wrt. second argument form of V, W. One can check that $\operatorname{rightker} f$ is non empty.

Let K be an add-associative right zeroed right complementable non empty double loop structure, let V be a right zeroed non empty vector space structure over K, and let f be an additive wrt. second argument form of V, V. Note that $\operatorname{diagker} f$ is non empty.

Let K be an add-associative right zeroed right complementable non empty double loop structure, let V be a right zeroed non empty vector space structure over K, and let f be an additive wrt. first argument form of V, V. Note that $\operatorname{diagker} f$ is non empty.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K, and let f be a homogeneous wrt. second argument form of V, V. One can check that $\operatorname{diagker} f$ is non empty.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K, and let f be a homogeneous wrt. first argument form of V, V. One can check that $\operatorname{diagker} f$ is non empty.

We now state three propositions:

(42) Let K be a zero structure, V be a non empty vector space structure over K, and f be a form of V, V. Then $\operatorname{leftker} f \subseteq \operatorname{diagker} f$ and $\operatorname{rightker} f \subseteq \operatorname{diagker} f$.

(43) Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, V, W be non empty vector space structures over K, and f be an additive wrt. first argument homogeneous wrt. first argument form of V, W. Then $\operatorname{leftker} f$ is linearly closed.

(44) Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, V, W be non empty vector space structures over K, and f be an additive wrt. second argument homogeneous wrt. second argument form of V, W. Then $\operatorname{rightker} f$ is linearly closed.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, let W be a non empty vector space structure over K, and let f be an additive wrt. first argument homogeneous wrt. first argument form of V, W. The functor $\operatorname{LKer} f$ yielding a strict non empty subspace of V is defined by:

(Def. 19) The carrier of $\operatorname{LKer} f = \operatorname{leftker} f$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a non empty vector space structure over K, let W be a vector space over K, and let f be an additive wrt. second argument homogeneous wrt. second argument form of V, W. The functor $\operatorname{RKer} f$ yielding a strict non empty subspace of W is defined by:

(Def. 20) The carrier of $\operatorname{RKer} f = \operatorname{rightker} f$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, let W be a non empty vector space structure over K, and let f be an additive wrt. first argument homogeneous wrt. first argument form of V, W. The functor LQForm(f) yields an additive wrt. first argument homogeneous wrt. first argument form of ${}^{V}/_{\mathrm{LKer}\,f}$, W and is defined by the condition (Def. 21).

(Def. 21) Let A be a vector of ${}^{V}/_{\mathrm{LKer}\,f}$, w be a vector of W, and v be a vector of V. If $A = v + \mathrm{LKer}\,f$, then $(\mathrm{LQForm}(f))(\langle A, w\rangle) = f(\langle v, w\rangle)$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a non empty vector space structure over K, let W be a vector space over K, and let f be an additive wrt. second argument homogeneous wrt. second argument form of V, W. The functor RQForm(f) yielding an additive wrt. second argument homogeneous wrt. second argument form of V, ${}^{W}/_{\mathrm{RKer}\,f}$ is defined by the condition (Def. 22).

(Def. 22) Let A be a vector of ${}^{W}/_{\mathrm{RKer}\,f}$, v be a vector of V, and w be a vector of W. If $A = w + \mathrm{RKer}\,f$, then $(\mathrm{RQForm}(f))(\langle v, A\rangle) = f(\langle v, w\rangle)$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V, W be vector spaces over K, and let f be a bilinear form of V, W. Note that LQForm(f) is additive wrt. second argument and homogeneous wrt. second argument and RQForm(f) is additive wrt. first argument and homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V, W be vector spaces over K, and let f be a bilinear form of V, W. The functor QForm(f) yields a bilinear form of ${}^{V}/_{\mathrm{LKer}\,f}$, ${}^{W}/_{\mathrm{RKer}\,f}$ and is defined by the condition (Def. 23).

(Def. 23) Let A be a vector of ${}^{V}/_{\mathrm{LKer}\,f}$, B be a vector of ${}^{W}/_{\mathrm{RKer}\,f}$, v be a vector of V, and w be a vector of W. If $A = v + \mathrm{LKer}\,f$ and $B = w + \mathrm{RKer}\,f$, then $(\mathrm{QForm}(f))(\langle A, B\rangle) = f(\langle v, w\rangle)$.

One can prove the following propositions:

(45) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a non empty vector space structure over K, and f be an additive wrt. first argument homogeneous wrt. first argument form of V, W. Then $\mathrm{rightker}\,f = \mathrm{rightker}(\mathrm{LQForm}(f))$.

(46) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a non empty vector space structure over K, W be a vector space over K, and f be an additive wrt. second argument homogeneous wrt. second argument form of V, W. Then $\mathrm{leftker}\,f = \mathrm{leftker}(\mathrm{RQForm}(f))$.

(47) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, and f be a bilinear form of V, W. Then $\mathrm{RKer}\, f = \mathrm{RKer}(\mathrm{LQForm}(f))$.

(48) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, and f be a bilinear form of V, W. Then $\mathrm{LKer}\, f = \mathrm{LKer}(\mathrm{RQForm}(f))$.

(49) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, and f be a bilinear form of V, W. Then $\mathrm{QForm}(f) = \mathrm{RQForm}(\mathrm{LQForm}(f))$ and $\mathrm{QForm}(f) = \mathrm{LQForm}(\mathrm{RQForm}(f))$.

(50) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, and f be a bilinear form of V, W. Then $\mathrm{leftker}(\mathrm{QForm}(f)) = \mathrm{leftker}(\mathrm{RQForm}(\mathrm{LQForm}(f)))$ and $\mathrm{rightker}(\mathrm{QForm}(f)) = \mathrm{rightker}(\mathrm{RQForm}(\mathrm{LQForm}(f)))$ and $\mathrm{leftker}(\mathrm{QForm}(f)) = \mathrm{leftker}(\mathrm{LQForm}(\mathrm{RQForm}(f)))$ and $\mathrm{rightker}(\mathrm{QForm}(f)) = \mathrm{rightker}(\mathrm{LQForm}(\mathrm{RQForm}(f)))$.

(51) Let K be an add-associative right zeroed right complementable distributive non empty double loop structure, V, W be non empty vector space structures over K, f be a functional in V, and g be a functional in W. Then $\ker f \subseteq \mathrm{leftker}(f \otimes g)$.

(52) Let K be an add-associative right zeroed right complementable associative commutative left unital field-like distributive non empty double loop structure, V, W be non empty vector space structures over K, f be a functional in V, and g be a functional in W. If $g \neq 0\mathrm{Functional}\, W$, then $\mathrm{leftker}(f \otimes g) = \ker f$.

(53) Let K be an add-associative right zeroed right complementable distributive non empty double loop structure, V, W be non empty vector space structures over K, f be a functional in V, and g be a functional in W. Then $\ker g \subseteq \mathrm{rightker}(f \otimes g)$.

(54) Let K be an add-associative right zeroed right complementable associative commutative left unital field-like distributive non empty double loop structure, V, W be non empty vector space structures over K, f be a functional in V, and g be a functional in W. If $f \neq 0\mathrm{Functional}\, V$, then $\mathrm{rightker}(f \otimes g) = \ker g$.

(55) Let K be an add-associative right zeroed right complementable commutative Abelian associative left unital distributive field-like non empty double loop structure, V be a vector space over K, W be a non empty

vector space structure over K, f be a linear functional in V, and g be a functional in W. If $g \neq 0\mathrm{Functional}\, W$, then $\mathrm{LKer}(f \otimes g) = \mathrm{Ker}\, f$ and $\mathrm{LQForm}(f \otimes g) = (\mathrm{CQFunctional}\, f) \otimes g$.

(56) Let K be an add-associative right zeroed right complementable commutative Abelian associative left unital distributive field-like non empty double loop structure, V be a non empty vector space structure over K, W be a vector space over K, f be a functional in V, and g be a linear functional in W. If $f \neq 0\mathrm{Functional}\, V$, then $\mathrm{RKer}(f \otimes g) = \mathrm{Ker}\, g$ and $\mathrm{RQForm}(f \otimes g) = f \otimes (\mathrm{CQFunctional}\, g)$.

(57) Let K be a field, V, W be non trivial vector spaces over K, f be a non constant linear functional in V, and g be a non constant linear functional in W. Then $\mathrm{QForm}(f \otimes g) = (\mathrm{CQFunctional}\, f) \otimes (\mathrm{CQFunctional}\, g)$.

Let K be a zero structure, let V, W be non empty vector space structures over K, and let f be a form of V, W. We say that f is degenerated on left if and only if:

(Def. 24) $\mathrm{leftker}\, f \neq \{0_V\}$.

We say that f is degenerated on right if and only if:

(Def. 25) $\mathrm{rightker}\, f \neq \{0_W\}$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, let W be a right zeroed non empty vector space structure over K, and let f be an additive wrt. first argument homogeneous wrt. first argument form of V, W. Note that $\mathrm{LQForm}(f)$ is non degenerated on left.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a right zeroed non empty vector space structure over K, let W be a vector space over K, and let f be an additive wrt. second argument homogeneous wrt. second argument form of V, W. Note that $\mathrm{RQForm}(f)$ is non degenerated on right.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V, W be vector spaces over K, and let f be a bilinear form of V, W. Observe that $\mathrm{QForm}(f)$ is non degenerated on left and non degenerated on right.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V, W be vector spaces over K, and let f be a bilinear form of V, W. One can verify that $\mathrm{RQForm}(\mathrm{LQForm}(f))$ is non degenerated on left and non degenerated on right and $\mathrm{LQForm}(\mathrm{RQForm}(f))$ is non degenerated on left and non degenerated on right.

Let K be a field, let V, W be non trivial vector spaces over K, and let f be a non constant bilinear form of V, W. Note that $\mathrm{QForm}(f)$ is non constant.

6. Bilinear Symmetric and Alternating Forms

Let K be a 1-sorted structure, let V be a vector space structure over K, and let f be a form of V, V. We say that f is symmetric if and only if:

(Def. 26) For all vectors v, w of V holds $f(\langle v, w\rangle) = f(\langle w, v\rangle)$.

Let K be a zero structure, let V be a vector space structure over K, and let f be a form of V, V. We say that f is alternating if and only if:

(Def. 27) For every vector v of V holds $f(\langle v, v\rangle) = 0_K$.

Let K be a non empty zero structure and let V be a non empty vector space structure over K. Observe that NulForm(V, V) is symmetric and NulForm(V, V) is alternating.

Let K be a non empty zero structure and let V be a non empty vector space structure over K. Observe that there exists a form of V, V which is symmetric and there exists a form of V, V which is alternating.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure and let V be a non empty vector space structure over K. Note that there exists a form of V, V which is symmetric, additive wrt. second argument, homogeneous wrt. second argument, additive wrt. first argument, and homogeneous wrt. first argument and there exists a form of V, V which is alternating, additive wrt. second argument, homogeneous wrt. second argument, additive wrt. first argument, and homogeneous wrt. first argument.

Let K be a field and let V be a non trivial vector space over K. Observe that there exists a form of V, V which is symmetric, non trivial, non constant, additive wrt. second argument, homogeneous wrt. second argument, additive wrt. first argument, and homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable non empty loop structure and let V be a non empty vector space structure over K. Note that there exists a form of V, V which is alternating, additive wrt. second argument, and additive wrt. first argument.

Let K be a non empty loop structure, let V be a non empty vector space structure over K, and let f, g be symmetric forms of V, V. One can check that $f + g$ is symmetric.

Let K be a non empty double loop structure, let V be a non empty vector space structure over K, let f be a symmetric form of V, V, and let a be an element of the carrier of K. One can check that $a \cdot f$ is symmetric.

Let K be a non empty loop structure, let V be a non empty vector space structure over K, and let f be a symmetric form of V, V. Note that $-f$ is symmetric.

Let K be a non empty loop structure, let V be a non empty vector space structure over K, and let f, g be symmetric forms of V, V. Observe that $f - g$

is symmetric.

Let K be a right zeroed non empty loop structure, let V be a non empty vector space structure over K, and let f, g be alternating forms of V, V. One can check that $f+g$ is alternating.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V be a non empty vector space structure over K, let f be an alternating form of V, V, and let a be a scalar of K. One can verify that $a \cdot f$ is alternating.

Let K be an add-associative right zeroed right complementable non empty loop structure, let V be a non empty vector space structure over K, and let f be an alternating form of V, V. Note that $-f$ is alternating.

Let K be an add-associative right zeroed right complementable non empty loop structure, let V be a non empty vector space structure over K, and let f, g be alternating forms of V, V. Observe that $f-g$ is alternating.

One can prove the following two propositions:

(58) Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, V be a non empty vector space structure over K, and f be a symmetric bilinear form of V, V. Then $\mathrm{leftker}\, f = \mathrm{rightker}\, f$.

(59) Let K be an add-associative right zeroed right complementable non empty loop structure, V be a non empty vector space structure over K, f be an alternating additive wrt. first argument additive wrt. second argument form of V, V, and v, w be vectors of V. Then $f(\langle v, w\rangle) = -f(\langle w, v\rangle)$.

Let K be a Fanoian field, let V be a non empty vector space structure over K, and let f be an additive wrt. first argument additive wrt. second argument form of V, V. Let us observe that f is alternating if and only if:

(Def. 28) For all vectors v, w of V holds $f(\langle v, w\rangle) = -f(\langle w, v\rangle)$.

Next we state the proposition

(60) Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, V be a non empty vector space structure over K, and f be an alternating bilinear form of V, V. Then $\mathrm{leftker}\, f = \mathrm{rightker}\, f$.

References

[1] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(**3**):537–541, 1990.

[2] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.

[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[7] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[8] Jarosław Kotowicz. Quotient vector spaces and functionals. *Formalized Mathematics*, 11(**1**):59–68, 2003.
[9] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[10] Anna Justyna Milewska. The Hahn Banach theorem in the vector space over the field of complex numbers. *Formalized Mathematics*, 9(**2**):363–371, 2001.
[11] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[12] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[14] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[15] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Formalized Mathematics*, 1(**5**):865–870, 1990.
[16] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[19] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received November 5, 2002