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Properties of the Upper and Lower Sequence on the Cage[1]

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The terminology and notation used here are introduced in the following articles: [24], [27], [1], [3], [4], [2], [14], [12], [25], [22], [23], [11], [21], [8], [9], [6], [26], [15], [10], [18], [17], [5], [20], [19], [7], [13], and [16].

In this paper n is a natural number.

We now state a number of propositions:

(1) For all subsets A, B of $\mathcal{E}^2_{\rm T}$ such that A meets B holds $\mathrm{proj1}^\circ A$ meets $\mathrm{proj1}^\circ B$.

(2) Let A, B be subsets of $\mathcal{E}^2_{\rm T}$ and s be a real number. If A misses B and $A \subseteq \mathrm{HorizontalLine}\, s$ and $B \subseteq \mathrm{HorizontalLine}\, s$, then $\mathrm{proj1}^\circ A$ misses $\mathrm{proj1}^\circ B$.

(3) For every closed subset S of $\mathcal{E}^2_{\rm T}$ such that S is Bounded holds $\mathrm{proj1}^\circ S$ is closed.

(4) For every compact subset S of $\mathcal{E}^2_{\rm T}$ holds $\mathrm{proj1}^\circ S$ is compact.

(5) Let p, q, p_1, q_1 be points of $\mathcal{E}^2_{\rm T}$. Suppose $\mathcal{L}(p, q)$ is vertical and $\mathcal{L}(p_1, q_1)$ is vertical and $p_{\bf 1} = (p_1)_{\bf 1}$ and $p_{\bf 2} \leqslant (p_1)_{\bf 2}$ and $(p_1)_{\bf 2} \leqslant (q_1)_{\bf 2}$ and $(q_1)_{\bf 2} \leqslant q_{\bf 2}$. Then $\mathcal{L}(p_1, q_1) \subseteq \mathcal{L}(p, q)$.

(6) Let p, q, p_1, q_1 be points of $\mathcal{E}^2_{\rm T}$. Suppose $\mathcal{L}(p, q)$ is horizontal and $\mathcal{L}(p_1, q_1)$ is horizontal and $p_{\bf 2} = (p_1)_{\bf 2}$ and $p_{\bf 1} \leqslant (p_1)_{\bf 1}$ and $(p_1)_{\bf 1} \leqslant (q_1)_{\bf 1}$ and $(q_1)_{\bf 1} \leqslant q_{\bf 1}$. Then $\mathcal{L}(p_1, q_1) \subseteq \mathcal{L}(p, q)$.

(7) Let G be a Go-board and i, j, k, j_1, k_1 be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \mathrm{len}\, G$ and $1 \leqslant j$ and $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $k \leqslant \mathrm{width}\, G$. Then $\mathcal{L}(G \circ (i, j_1), G \circ (i, k_1)) \subseteq \mathcal{L}(G \circ (i, j), G \circ (i, k))$.

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(8) Let G be a Go-board and i, j, k, j_1, k_1 be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \operatorname{width} G$ and $1 \leqslant j$ and $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $k \leqslant \operatorname{len} G$. Then $\mathcal{L}(G \circ (j_1, i), G \circ (k_1, i)) \subseteq \mathcal{L}(G \circ (j, i), G \circ (k, i))$.

(9) Let G be a Go-board and j, k, j_1, k_1 be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $k \leqslant \operatorname{width} G$. Then $\mathcal{L}(G \circ (\operatorname{Center} G, j_1), G \circ (\operatorname{Center} G, k_1)) \subseteq \mathcal{L}(G \circ (\operatorname{Center} G, j), G \circ (\operatorname{Center} G, k))$.

(10) Let G be a Go-board. Suppose $\operatorname{len} G = \operatorname{width} G$. Let j, k, j_1, k_1 be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $k \leqslant \operatorname{len} G$. Then $\mathcal{L}(G \circ (j_1, \operatorname{Center} G), G \circ (k_1, \operatorname{Center} G)) \subseteq \mathcal{L}(G \circ (j, \operatorname{Center} G), G \circ (k, \operatorname{Center} G))$.

(11) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (i, j) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n))$. Then there exists a natural number j_1 such that $j \leqslant j_1$ and $j_1 \leqslant k$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i, j_1), \operatorname{Gauge}(C, n) \circ (i, k)) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (i, j_1)\}$.

(12) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (i, k) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n))$. Then there exists a natural number k_1 such that $j \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i, j), \operatorname{Gauge}(C, n) \circ (i, k_1)) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (i, k_1)\}$.

(13) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (i, j) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n))$ and $\operatorname{Gauge}(C, n) \circ (i, k) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n))$. Then there exist natural numbers j_1, k_1 such that $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i, j_1), \operatorname{Gauge}(C, n) \circ (i, k_1)) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (i, j_1)\}$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (i, j_1), \operatorname{Gauge}(C, n) \circ (i, k_1)) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (i, k_1)\}$.

(14) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (j, i) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n))$. Then there exists a natural number j_1 such that $j \leqslant j_1$ and $j_1 \leqslant k$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j_1, i), \operatorname{Gauge}(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (j_1, i)\}$.

(15) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (k, i) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n))$. Then there exists a natural number k_1 such that $j \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k_1, i)) \cap$

$\widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (k_1,i)\}$.

(16) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant i$ and $i \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $\mathrm{Gauge}(C,n) \circ (j,i) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$ and $\mathrm{Gauge}(C,n) \circ (k,i) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$. Then there exist natural numbers j_1, k_1 such that $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (j_1,i), \mathrm{Gauge}(C,n) \circ (k_1,i)) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (j_1,i)\}$ and $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (j_1,i), \mathrm{Gauge}(C,n) \circ (k_1,i)) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (k_1,i)\}$.

(17) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $\mathrm{Gauge}(C,n) \circ (i,j) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$. Then there exists a natural number j_1 such that $j \leqslant j_1$ and $j_1 \leqslant k$ and $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,j_1), \mathrm{Gauge}(C,n) \circ (i,k)) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i,j_1)\}$.

(18) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $\mathrm{Gauge}(C,n) \circ (i,k) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$. Then there exists a natural number k_1 such that $j \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,j), \mathrm{Gauge}(C,n) \circ (i,k_1)) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i,k_1)\}$.

(19) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant i$ and $i \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $\mathrm{Gauge}(C,n) \circ (i,j) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$ and $\mathrm{Gauge}(C,n) \circ (i,k) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$. Then there exist natural numbers j_1, k_1 such that $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,j_1), \mathrm{Gauge}(C,n) \circ (i,k_1)) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i,j_1)\}$ and $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i,j_1), \mathrm{Gauge}(C,n) \circ (i,k_1)) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i,k_1)\}$.

(20) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant i$ and $i \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $\mathrm{Gauge}(C,n) \circ (j,i) \in \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n))$. Then there exists a natural number j_1 such that $j \leqslant j_1$ and $j_1 \leqslant k$ and $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (j_1,i), \mathrm{Gauge}(C,n) \circ (k,i)) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (j_1,i)\}$.

(21) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant i$ and $i \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $\mathrm{Gauge}(C,n) \circ (k,i) \in \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n))$. Then there exists a natural number k_1 such that $j \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (j,i), \mathrm{Gauge}(C,n) \circ (k_1,i)) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (k_1,i)\}$.

(22) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\mathrm{T}}$ and i, j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{len} \operatorname{Gauge}(C,n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C,n)$ and $\operatorname{Gauge}(C,n) \circ (j,i) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n))$ and $\operatorname{Gauge}(C,n) \circ (k,i) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n))$. Then there exist natural numbers j_1, k_1 such that $j \leqslant j_1$ and $j_1 \leqslant k_1$ and $k_1 \leqslant k$ and $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (j_1,i), \operatorname{Gauge}(C,n) \circ (k_1,i)) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n)) = \{\operatorname{Gauge}(C,n) \circ (j_1,i)\}$ and $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (j_1,i), \operatorname{Gauge}(C,n) \circ (k_1,i)) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n)) = \{\operatorname{Gauge}(C,n) \circ (k_1,i)\}$.

(23) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \operatorname{len} \operatorname{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \operatorname{Gauge}(C,n)$ and $\operatorname{Gauge}(C,n) \circ (i,k) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n))$ and $\operatorname{Gauge}(C,n) \circ (i,j) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n))$. Then $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i,j), \operatorname{Gauge}(C,n) \circ (i,k))$ meets $\operatorname{LowerArc} C$.

(24) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \operatorname{len} \operatorname{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \operatorname{Gauge}(C,n)$ and $\operatorname{Gauge}(C,n) \circ (i,k) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C,n))$ and $\operatorname{Gauge}(C,n) \circ (i,j) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C,n))$. Then $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i,j), \operatorname{Gauge}(C,n) \circ (i,k))$ meets $\operatorname{UpperArc} C$.

(25) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \operatorname{len} \operatorname{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \operatorname{Gauge}(C,n)$ and $n > 0$ and $\operatorname{Gauge}(C,n) \circ (i,k) \in \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$ and $\operatorname{Gauge}(C,n) \circ (i,j) \in \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$. Then $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i,j), \operatorname{Gauge}(C,n) \circ (i,k))$ meets $\operatorname{LowerArc} C$.

(26) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \operatorname{len} \operatorname{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \operatorname{Gauge}(C,n)$ and $n > 0$ and $\operatorname{Gauge}(C,n) \circ (i,k) \in \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$ and $\operatorname{Gauge}(C,n) \circ (i,j) \in \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n))$. Then $\mathcal{L}(\operatorname{Gauge}(C,n) \circ (i,j), \operatorname{Gauge}(C,n) \circ (i,k))$ meets $\operatorname{UpperArc} C$.

(27) Let C be a simple closed curve and j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \operatorname{Gauge}(C,n+1)$ and $\operatorname{Gauge}(C,n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C,n+1),k) \in \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n+1))$ and $\operatorname{Gauge}(C,n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C,n+1),j) \in \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n+1))$. Then $\mathcal{L}(\operatorname{Gauge}(C,n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C,n+1),j), \operatorname{Gauge}(C,n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C,n+1),k))$ meets $\operatorname{LowerArc} C$.

(28) Let C be a simple closed curve and j, k be natural numbers. Suppose $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} \operatorname{Gauge}(C,n+1)$ and $\operatorname{Gauge}(C,n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C,n+1),k) \in \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n+1))$ and $\operatorname{Gauge}(C,n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C,n+1),j) \in \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C,n+1))$. Then $\mathcal{L}(\operatorname{Gauge}(C,n+1) \circ (\operatorname{Center} \operatorname{Gauge}(C,n+1),j), \operatorname{Gauge}(C,n+1) \circ$

(Center Gauge$(C, n+1), k)$) meets UpperArc C.

(29) Let C be a compact connected non vertical non horizontal subset of $\mathcal{E}^2_{\rm T}$ and i, j, k be natural numbers. Suppose $1 < j$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (k, i) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n))$ and $\operatorname{Gauge}(C, n) \circ (j, i) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n))$. Then $j \neq k$.

(30) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (k, i)\}$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (j, i)\}$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets LowerArc C.

(31) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (k, i)\}$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (j, i)\}$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets UpperArc C.

(32) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (k, i) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n))$ and $\operatorname{Gauge}(C, n) \circ (j, i) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets LowerArc C.

(33) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (k, i) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n))$ and $\operatorname{Gauge}(C, n) \circ (j, i) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets UpperArc C.

(34) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $n > 0$ and $\operatorname{Gauge}(C, n) \circ (k, i) \in \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$ and $\operatorname{Gauge}(C, n) \circ (j, i) \in \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets LowerArc C.

(35) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $n > 0$ and $\operatorname{Gauge}(C, n) \circ (k, i) \in \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$ and $\operatorname{Gauge}(C, n) \circ (j, i) \in \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets UpperArc C.

(36) Let C be a simple closed curve and j, k be natural numbers. Sup-

pose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n+1)$ and $\operatorname{Gauge}(C, n+1) \circ (k, \operatorname{Center} \operatorname{Gauge}(C, n+1)) \in \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1))$ and $\operatorname{Gauge}(C, n+1) \circ (j, \operatorname{Center} \operatorname{Gauge}(C, n+1)) \in \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n+1) \circ (j, \operatorname{Center} \operatorname{Gauge}(C, n+1)), \operatorname{Gauge}(C, n+1) \circ (k, \operatorname{Center} \operatorname{Gauge}(C, n+1)))$ meets $\operatorname{LowerArc} C$.

(37) Let C be a simple closed curve and j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n+1)$ and $\operatorname{Gauge}(C, n+1) \circ (k, \operatorname{Center} \operatorname{Gauge}(C, n+1)) \in \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1))$ and $\operatorname{Gauge}(C, n+1) \circ (j, \operatorname{Center} \operatorname{Gauge}(C, n+1)) \in \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n+1))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n+1) \circ (j, \operatorname{Center} \operatorname{Gauge}(C, n+1)), \operatorname{Gauge}(C, n+1) \circ (k, \operatorname{Center} \operatorname{Gauge}(C, n+1)))$ meets $\operatorname{UpperArc} C$.

(38) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (j, i)\}$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (k, i)\}$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets $\operatorname{LowerArc} C$.

(39) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (j, i)\}$ and $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i)) \cap \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n)) = \{\operatorname{Gauge}(C, n) \circ (k, i)\}$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets $\operatorname{UpperArc} C$.

(40) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (j, i) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n))$ and $\operatorname{Gauge}(C, n) \circ (k, i) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets $\operatorname{LowerArc} C$.

(41) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $\operatorname{Gauge}(C, n) \circ (j, i) \in \widetilde{\mathcal{L}}(\operatorname{UpperSeq}(C, n))$ and $\operatorname{Gauge}(C, n) \circ (k, i) \in \widetilde{\mathcal{L}}(\operatorname{LowerSeq}(C, n))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets $\operatorname{UpperArc} C$.

(42) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$ and $1 \leqslant i$ and $i \leqslant \operatorname{width} \operatorname{Gauge}(C, n)$ and $n > 0$ and $\operatorname{Gauge}(C, n) \circ (j, i) \in \operatorname{UpperArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$ and $\operatorname{Gauge}(C, n) \circ (k, i) \in \operatorname{LowerArc} \widetilde{\mathcal{L}}(\operatorname{Cage}(C, n))$. Then $\mathcal{L}(\operatorname{Gauge}(C, n) \circ (j, i), \operatorname{Gauge}(C, n) \circ (k, i))$ meets $\operatorname{LowerArc} C$.

(43) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \operatorname{len} \operatorname{Gauge}(C, n)$

and $1 \leqslant i$ and $i \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $n > 0$ and $\mathrm{Gauge}(C,n) \circ (j,i) \in \mathrm{UpperArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$ and $\mathrm{Gauge}(C,n) \circ (k,i) \in \mathrm{LowerArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n))$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (j,i), \mathrm{Gauge}(C,n) \circ (k,i))$ meets $\mathrm{UpperArc}\,C$.

(44) Let C be a simple closed curve and j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \mathrm{len}\,\mathrm{Gauge}(C,n+1)$ and $\mathrm{Gauge}(C,n+1) \circ (j, \mathrm{Center}\,\mathrm{Gauge}(C,n+1)) \in \mathrm{UpperArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1))$ and $\mathrm{Gauge}(C,n+1) \circ (k, \mathrm{Center}\,\mathrm{Gauge}(C,n+1)) \in \mathrm{LowerArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1))$. Then $\mathcal{L}(\mathrm{Gauge}(C,n+1) \circ (j, \mathrm{Center}\,\mathrm{Gauge}(C,n+1)), \mathrm{Gauge}(C,n+1) \circ (k, \mathrm{Center}\,\mathrm{Gauge}(C,n+1)))$ meets $\mathrm{LowerArc}\,C$.

(45) Let C be a simple closed curve and j, k be natural numbers. Suppose $1 < j$ and $j \leqslant k$ and $k < \mathrm{len}\,\mathrm{Gauge}(C,n+1)$ and $\mathrm{Gauge}(C,n+1) \circ (j, \mathrm{Center}\,\mathrm{Gauge}(C,n+1)) \in \mathrm{UpperArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1))$ and $\mathrm{Gauge}(C,n+1) \circ (k, \mathrm{Center}\,\mathrm{Gauge}(C,n+1)) \in \mathrm{LowerArc}\,\widetilde{\mathcal{L}}(\mathrm{Cage}(C,n+1))$. Then $\mathcal{L}(\mathrm{Gauge}(C,n+1) \circ (j, \mathrm{Center}\,\mathrm{Gauge}(C,n+1)), \mathrm{Gauge}(C,n+1) \circ (k, \mathrm{Center}\,\mathrm{Gauge}(C,n+1)))$ meets $\mathrm{UpperArc}\,C$.

(46) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 \leqslant i_2$ and $i_2 < \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $(\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i_2,k)\}$ and $(\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i_1,j)\}$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))$ meets $\mathrm{UpperArc}\,C$.

(47) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 \leqslant i_2$ and $i_2 < \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $(\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i_2,k)\}$ and $(\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i_1,j)\}$. Then $\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))$ meets $\mathrm{LowerArc}\,C$.

(48) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_2$ and $i_2 \leqslant i_1$ and $i_1 < \mathrm{len}\,\mathrm{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \mathrm{width}\,\mathrm{Gauge}(C,n)$ and $(\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))) \cap \widetilde{\mathcal{L}}(\mathrm{UpperSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i_2,k)\}$ and $(\mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,j), \mathrm{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\mathrm{Gauge}(C,n) \circ (i_1,k), \mathrm{Gauge}(C,n) \circ (i_2,k))) \cap \widetilde{\mathcal{L}}(\mathrm{LowerSeq}(C,n)) = \{\mathrm{Gauge}(C,n) \circ (i_1,j)\}$.

Then $\mathcal{L}(\text{Gauge}(C,n) \circ (i_1,j), \text{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\text{Gauge}(C,n) \circ (i_1,k), \text{Gauge}(C,n) \circ (i_2,k))$ meets $\text{UpperArc}\, C$.

(49) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_2$ and $i_2 \leqslant i_1$ and $i_1 < \text{len}\,\text{Gauge}(C,n)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \text{width}\,\text{Gauge}(C,n)$ and $(\mathcal{L}(\text{Gauge}(C,n) \circ (i_1,j), \text{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\text{Gauge}(C,n) \circ (i_1,k), \text{Gauge}(C,n)\circ(i_2,k))) \cap \widetilde{\mathcal{L}}(\text{UpperSeq}(C,n)) = \{\text{Gauge}(C,n)\circ(i_2,k)\}$ and $(\mathcal{L}(\text{Gauge}(C,n) \circ (i_1,j), \text{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\text{Gauge}(C,n) \circ (i_1,k), \text{Gauge}(C,n)\circ(i_2,k))) \cap \widetilde{\mathcal{L}}(\text{LowerSeq}(C,n)) = \{\text{Gauge}(C,n)\circ(i_1,j)\}$. Then $\mathcal{L}(\text{Gauge}(C,n) \circ (i_1,j), \text{Gauge}(C,n) \circ (i_1,k)) \cup \mathcal{L}(\text{Gauge}(C,n) \circ (i_1,k), \text{Gauge}(C,n) \circ (i_2,k))$ meets $\text{LowerArc}\, C$.

(50) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 < \text{len}\,\text{Gauge}(C,n+1)$ and $1 < i_2$ and $i_2 < \text{len}\,\text{Gauge}(C,n+1)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \text{width}\,\text{Gauge}(C,n+1)$ and $\text{Gauge}(C,n+1) \circ (i_1,k) \in \text{UpperArc}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n+1))$ and $\text{Gauge}(C,n+1) \circ (i_2,j) \in \text{LowerArc}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n+1))$. Then $\mathcal{L}(\text{Gauge}(C,n+1) \circ (i_2,j), \text{Gauge}(C,n+1) \circ (i_2,k)) \cup \mathcal{L}(\text{Gauge}(C,n+1) \circ (i_2,k), \text{Gauge}(C,n+1) \circ (i_1,k))$ meets $\text{UpperArc}\, C$.

(51) Let C be a simple closed curve and i_1, i_2, j, k be natural numbers. Suppose that $1 < i_1$ and $i_1 < \text{len}\,\text{Gauge}(C,n+1)$ and $1 < i_2$ and $i_2 < \text{len}\,\text{Gauge}(C,n+1)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \text{width}\,\text{Gauge}(C,n+1)$ and $\text{Gauge}(C,n+1) \circ (i_1,k) \in \text{UpperArc}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n+1))$ and $\text{Gauge}(C,n+1) \circ (i_2,j) \in \text{LowerArc}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n+1))$. Then $\mathcal{L}(\text{Gauge}(C,n+1) \circ (i_2,j), \text{Gauge}(C,n+1) \circ (i_2,k)) \cup \mathcal{L}(\text{Gauge}(C,n+1) \circ (i_2,k), \text{Gauge}(C,n+1) \circ (i_1,k))$ meets $\text{LowerArc}\, C$.

(52) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \text{len}\,\text{Gauge}(C,n+1)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \text{width}\,\text{Gauge}(C,n+1)$ and $\text{Gauge}(C,n+1) \circ (i,k) \in \text{UpperArc}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n+1))$ and $\text{Gauge}(C,n+1) \circ (\text{Center}\,\text{Gauge}(C,n+1),j) \in \text{LowerArc}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n+1))$. Then $\mathcal{L}(\text{Gauge}(C,n+1) \circ (\text{Center}\,\text{Gauge}(C,n+1),j), \text{Gauge}(C,n+1)\circ(\text{Center}\,\text{Gauge}(C,n+1),k)) \cup \mathcal{L}(\text{Gauge}(C,n+1) \circ (\text{Center}\,\text{Gauge}(C,n+1),k), \text{Gauge}(C,n+1) \circ (i,k))$ meets $\text{UpperArc}\, C$.

(53) Let C be a simple closed curve and i, j, k be natural numbers. Suppose $1 < i$ and $i < \text{len}\,\text{Gauge}(C,n+1)$ and $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \text{width}\,\text{Gauge}(C,n+1)$ and $\text{Gauge}(C,n+1) \circ (i,k) \in \text{UpperArc}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n+1))$ and $\text{Gauge}(C,n+1) \circ (\text{Center}\,\text{Gauge}(C,n+1),j) \in \text{LowerArc}\,\widetilde{\mathcal{L}}(\text{Cage}(C,n+1))$. Then $\mathcal{L}(\text{Gauge}(C,n+1) \circ (\text{Center}\,\text{Gauge}(C,n+1),j), \text{Gauge}(C,n+1)\circ(\text{Center}\,\text{Gauge}(C,n+1),k)) \cup \mathcal{L}(\text{Gauge}(C,n+1) \circ (\text{Center}\,\text{Gauge}(C,n+1),k), \text{Gauge}(C,n+1) \circ (i,k))$ meets $\text{LowerArc}\, C$.

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