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On the Decompositions of Intervals and Simple Closed Curves[1]

Adam Grabowski
University of Białystok

Summary. The aim of the paper is to show that the only subcontinua of the Jordan curve are arcs, the whole curve, and singletons of its points. Additionally, it has been shown that the only subcontinua of the unit interval $\mathbb{I}$ are closed intervals.

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The articles [21], [23], [13], [24], [2], [1], [3], [25], [19], [6], [4], [20], [8], [10], [11], [15], [26], [18], [22], [14], [16], [9], [17], [5], [12], and [7] provide the terminology and notation for this paper.

1. Preliminaries

Let us note that every simple closed curve is non trivial.

Let T be a non empty topological space. One can check that there exists a subset of T which is non empty, compact, and connected.

Let us observe that every element of the carrier of $\mathbb{I}$ is real.

Next we state two propositions:

(1) Let X be a non empty set and A, B be non empty subsets of X. If $A \subset B$, then there exists an element p of X such that $p \in B$ and $A \subseteq B \setminus \{p\}$.

(2) Let X be a non empty set and A be a non empty subset of X. Then A is trivial if and only if there exists an element x of X such that $A = \{x\}$.

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Let T be a non trivial 1-sorted structure. Observe that there exists a subset of the carrier of T which is non trivial.

The following proposition is true

(3) For every non trivial set X and for every set p there exists an element q of X such that $q \neq p$.

Let X be a non trivial set. Observe that there exists a subset of X which is non trivial.

We now state a number of propositions:

(4) Let T be a non trivial set, X be a non trivial subset of T, and p be a set. Then there exists an element q of T such that $q \in X$ and $q \neq p$.

(5) Let f, g be functions and a be a set. Suppose f is one-to-one and g is one-to-one and $\operatorname{dom} f \cap \operatorname{dom} g = \{a\}$ and $\operatorname{rng} f \cap \operatorname{rng} g = \{f(a)\}$. Then $f{+\cdot}g$ is one-to-one.

(6) Let f, g be functions and a be a set. Suppose f is one-to-one and g is one-to-one and $\operatorname{dom} f \cap \operatorname{dom} g = \{a\}$ and $\operatorname{rng} f \cap \operatorname{rng} g = \{f(a)\}$ and $f(a) = g(a)$. Then $(f{+\cdot}g)^{-1} = f^{-1}{+\cdot}g^{-1}$.

(7) Let n be a natural number, A be a non empty subset of the carrier of $\mathcal{E}^n_{\mathrm{T}}$, and p, q be points of $\mathcal{E}^n_{\mathrm{T}}$. If A is an arc from p to q, then $A \setminus \{p\}$ is non empty.

(8) For every natural number n and for all points a, b of $\mathcal{E}^n_{\mathrm{T}}$ holds $\mathcal{L}(a, b)$ is n-convex.

(9) For all real numbers s_1, s_3, s_4, l such that $s_1 \leqslant s_3$ and $s_1 < s_4$ and $0 \leqslant l$ and $l \leqslant 1$ holds $s_1 \leqslant (1 - l) \cdot s_3 + l \cdot s_4$.

(10) For every set x and for all real numbers a, b such that $a \leqslant b$ and $x \in [a, b]$ holds $x \in]a, b[$ or $x = a$ or $x = b$.

(11) For all real numbers a, b, c, d such that $]a, b[$ meets $[c, d]$ holds $b > c$.

(12) For all real numbers a, b, c, d such that $b \leqslant c$ holds $[a, b]$ misses $]c, d[$.

(13) For all real numbers a, b, c, d such that $b \leqslant c$ holds $]a, b[$ misses $[c, d]$.

(14) For all real numbers a, b, c, d such that $a < b$ and $[a, b] \subseteq [c, d]$ holds $c \leqslant a$ and $b \leqslant d$.

(15) For all real numbers a, b, c, d such that $a < b$ and $]a, b[\subseteq [c, d]$ holds $c \leqslant a$ and $b \leqslant d$.

(16) For all real numbers a, b, c, d such that $a < b$ and $]a, b[\subseteq [c, d]$ holds $[a, b] \subseteq [c, d]$.

(17) Let A be a subset of the carrier of $\mathbb{I}$ and a, b be real numbers. If $a < b$ and $A =]a, b[$, then $[a, b] \subseteq$ the carrier of $\mathbb{I}$.

(18) Let A be a subset of the carrier of $\mathbb{I}$ and a, b be real numbers. If $a < b$ and $A =]a, b]$, then $[a, b] \subseteq$ the carrier of $\mathbb{I}$.

(19) Let A be a subset of the carrier of $\mathbb{I}$ and a, b be real numbers. If $a < b$ and $A = [a, b[$, then $[a, b] \subseteq$ the carrier of $\mathbb{I}$.

(20) For all real numbers a, b such that $a \neq b$ holds $\overline{]a, b]} = [a, b]$.

(21) For all real numbers a, b such that $a \neq b$ holds $\overline{[a, b[} = [a, b]$.

(22) For every subset A of $\mathbb{I}$ and for all real numbers a, b such that $a < b$ and $A =]a, b[$ holds $\overline{A} = [a, b]$.

(23) For every subset A of the carrier of $\mathbb{I}$ and for all real numbers a, b such that $a < b$ and $A =]a, b]$ holds $\overline{A} = [a, b]$.

(24) For every subset A of the carrier of $\mathbb{I}$ and for all real numbers a, b such that $a < b$ and $A = [a, b[$ holds $\overline{A} = [a, b]$.

(25) For all real numbers a, b such that $a < b$ holds $[a, b] \neq]a, b]$.

(26) For all real numbers a, b holds $[a, b[$ misses $\{b\}$ and $]a, b]$ misses $\{a\}$.

(27) For all real numbers a, b such that $a \leqslant b$ holds $[a, b] \setminus \{a\} =]a, b]$.

(28) For all real numbers a, b such that $a \leqslant b$ holds $[a, b] \setminus \{b\} = [a, b[$.

(29) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $]a, b] \cap [b, c[= \{b\}$.

(30) For all real numbers a, b, c holds $[a, b[$ misses $[b, c]$ and $[a, b]$ misses $]b, c]$.

(31) For all real numbers a, b, c such that $a \leqslant b$ and $b \leqslant c$ holds $[a, c] \setminus \{b\} = [a, b[\cup]b, c]$.

(32) Let A be a subset of the carrier of $\mathbb{I}$ and a, b be real numbers. If $a \leqslant b$ and $A = [a, b]$, then $0 \leqslant a$ and $b \leqslant 1$.

(33) Let A, B be subsets of $\mathbb{I}$ and a, b, c be real numbers. If $a < b$ and $b < c$ and $A = [a, b[$ and $B =]b, c]$, then A and B are separated.

(34) For all real numbers a, b such that $a \leqslant b$ holds $[a, b] = [a, b[\cup \{b\}$.

(35) For all real numbers a, b such that $a \leqslant b$ holds $[a, b] = \{a\} \cup]a, b]$.

(36) For all real numbers a, b, c, d such that $a \leqslant b$ and $b < c$ and $c \leqslant d$ holds $[a, d] = [a, b] \cup]b, c[\cup [c, d]$.

(37) For all real numbers a, b, c, d such that $a \leqslant b$ and $b < c$ and $c \leqslant d$ holds $[a, d] \setminus ([a, b] \cup [c, d]) =]b, c[$.

(38) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $]a, b] \cup]b, c[=]a, c[$.

(39) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $[b, c[\subseteq]a, c[$.

(40) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $]a, b] \cup [b, c[=]a, c[$.

(41) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $]a, c[\setminus]a, b] =]b, c[$.

(42) For all real numbers a, b, c such that $a < b$ and $b < c$ holds $]a, c[\setminus [b, c[=]a, b[$.

(43) For all points p_1, p_2 of $\mathbb{I}$ holds $[p_1, p_2]$ is a subset of $\mathbb{I}$.

(44) For all points a, b of $\mathbb{I}$ holds $]a, b[$ is a subset of $\mathbb{I}$.

2. Decompositions of Intervals

The following propositions are true:

(45) For every real number p holds $\{p\}$ is a closed-interval subset of $\mathbb{R}$.

(46) Let A be a non empty connected subset of $\mathbb{I}$ and a, b, c be points of $\mathbb{I}$. If $a \leqslant b$ and $b \leqslant c$ and $a \in A$ and $c \in A$, then $b \in A$.

(47) For every non empty connected subset A of $\mathbb{I}$ and for all real numbers a, b such that $a \in A$ and $b \in A$ holds $[a, b] \subseteq A$.

(48) For all real numbers a, b and for every subset A of $\mathbb{I}$ such that $a \leqslant b$ and $A = [a, b]$ holds A is closed.

(49) For all points p_1, p_2 of $\mathbb{I}$ such that $p_1 \leqslant p_2$ holds $[p_1, p_2]$ is a non empty compact connected subset of $\mathbb{I}$.

(50) Let X be a subset of the carrier of $\mathbb{I}$ and X' be a subset of $\mathbb{R}$. If $X' = X$, then X' is upper bounded and lower bounded.

(51) Let X be a subset of the carrier of $\mathbb{I}$, X' be a subset of $\mathbb{R}$, and x be a real number. If $x \in X'$ and $X' = X$, then $\inf X' \leqslant x$ and $x \leqslant \sup X'$.

(52) For every subset A of $\mathbb{R}$ and for every subset B of $\mathbb{I}$ such that $A = B$ holds A is closed iff B is closed.

(53) For every closed-interval subset C of $\mathbb{R}$ holds $\inf C \leqslant \sup C$.

(54) Let C be a non empty compact connected subset of $\mathbb{I}$ and C' be a subset of $\mathbb{R}$. If $C = C'$ and $[\inf C', \sup C'] \subseteq C'$, then $[\inf C', \sup C'] = C'$.

(55) Every non empty compact connected subset of $\mathbb{I}$ is a closed-interval subset of $\mathbb{R}$.

(56) For every non empty compact connected subset C of $\mathbb{I}$ there exist points p_1, p_2 of $\mathbb{I}$ such that $p_1 \leqslant p_2$ and $C = [p_1, p_2]$.

3. Decompositions of Simple Closed Curves

The strict non empty subspace $I(01)$ of $\mathbb{I}$ is defined as follows:

(Def. 1) The carrier of $I(01) =]0, 1[$.

One can prove the following propositions:

(57) For every subset A of $\mathbb{I}$ such that $A =$ the carrier of $I(01)$ holds $I(01) = \mathbb{I} \restriction A$.

(58) The carrier of $I(01) =$ (the carrier of $\mathbb{I}$) $\setminus \{0, 1\}$.

(59) $I(01)$ is an open subspace of $\mathbb{I}$.

(60) For every real number r holds $r \in$ the carrier of $I(01)$ iff $0 < r$ and $r < 1$.

(61) For all points a, b of $\mathbb{I}$ such that $a < b$ and $b \neq 1$ holds $]a, b]$ is a non empty subset of $I(01)$.

(62) For all points a, b of $\mathbb{I}$ such that $a < b$ and $a \neq 0$ holds $[a, b[$ is a non empty subset of $I(01)$.

(63) For every simple closed curve D holds $(\mathcal{E}^2_{\mathrm{T}}) \restriction \square_{\mathcal{E}^2}$ and $(\mathcal{E}^2_{\mathrm{T}}) \restriction D$ are homeomorphic.

(64) Let D be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If D is an arc from p_1 to p_2, then $I(01)$ and $(\mathcal{E}^2_{\mathrm{T}}) \restriction (D \setminus \{p_1, p_2\})$ are homeomorphic.

(65) Let D be a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If D is an arc from p_1 to p_2, then $\mathbb{I}$ and $(\mathcal{E}^2_{\mathrm{T}}) \restriction D$ are homeomorphic.

(66) For all points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that $p_1 \neq p_2$ holds $\mathbb{I}$ and $(\mathcal{E}^2_{\mathrm{T}}) \restriction \mathcal{L}(p_1, p_2)$ are homeomorphic.

(67) Let E be a subset of $I(01)$. Given points p_1, p_2 of $\mathbb{I}$ such that $p_1 < p_2$ and $E = [p_1, p_2]$. Then $\mathbb{I}$ and $I(01) \restriction E$ are homeomorphic.

(68) Let A be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, p, q be points of $\mathcal{E}^2_{\mathrm{T}}$, and a, b be points of $\mathbb{I}$. Suppose A is an arc from p to q and $a < b$. Then there exists a non empty subset E of $\mathbb{I}$ and there exists a map f from $\mathbb{I} \restriction E$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction A$ such that $E = [a, b]$ and f is a homeomorphism and $f(a) = p$ and $f(b) = q$.

(69) Let A be a topological space, B be a non empty topological space, f be a map from A into B, C be a topological space, and X be a subset of A. Suppose f is continuous and C is a subspace of B. Let h be a map from $A \restriction X$ into C. If $h = f \restriction X$, then h is continuous.

(70) For every subset X of $\mathbb{I}$ and for all points a, b of $\mathbb{I}$ such that $a \leqslant b$ and $X =]a, b[$ holds X is open.

(71) For every subset X of $I(01)$ and for all points a, b of $\mathbb{I}$ such that $a \leqslant b$ and $X =]a, b[$ holds X is open.

(72) For every non empty subset X of $I(01)$ and for every point a of $\mathbb{I}$ such that $0 < a$ and $X =]0, a]$ holds X is closed.

(73) For every non empty subset X of $I(01)$ and for every point a of $\mathbb{I}$ such that $X = [a, 1[$ holds X is closed.

(74) Let A be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, p, q be points of $\mathcal{E}^2_{\mathrm{T}}$, and a, b be points of $\mathbb{I}$. Suppose A is an arc from p to q and $a < b$ and $b \neq 1$. Then there exists a non empty subset E of $I(01)$ and there exists a map f from $I(01) \restriction E$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction (A \setminus \{p\})$ such that $E =]a, b]$ and f is a homeomorphism and $f(b) = q$.

(75) Let A be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$, p, q be points of $\mathcal{E}^2_{\mathrm{T}}$, and a, b be points of $\mathbb{I}$. Suppose A is an arc from p to q and $a < b$ and $a \neq 0$. Then there exists a non empty subset E of $I(01)$ and there exists

a map f from $I(01)\restriction E$ into $(\mathcal{E}^2_{\mathrm{T}})\restriction(A \setminus \{q\})$ such that $E = [a, b[$ and f is a homeomorphism and $f(a) = p$.

(76) Let A, B be non empty subsets of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose A is an arc from p to q and B is an arc from q to p and $A \cap B = \{p, q\}$ and $p \neq q$. Then $I(01)$ and $(\mathcal{E}^2_{\mathrm{T}})\restriction((A \setminus \{p\}) \cup (B \setminus \{p\}))$ are homeomorphic.

(77) For every simple closed curve D and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in D$ holds $(\mathcal{E}^2_{\mathrm{T}})\restriction(D \setminus \{p\})$ and $I(01)$ are homeomorphic.

(78) Let D be a simple closed curve and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in D$ and $q \in D$, then $(\mathcal{E}^2_{\mathrm{T}})\restriction(D \setminus \{p\})$ and $(\mathcal{E}^2_{\mathrm{T}})\restriction(D \setminus \{q\})$ are homeomorphic.

(79) Let C be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and E be a subset of $I(01)$. Suppose there exist points p_1, p_2 of $\mathbb{I}$ such that $p_1 < p_2$ and $E = [p_1, p_2]$ and $I(01)\restriction E$ and $(\mathcal{E}^2_{\mathrm{T}})\restriction C$ are homeomorphic. Then there exist points s_1, s_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that C is an arc from s_1 to s_2.

(80) Let D_1 be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, f be a map from $(\mathcal{E}^2_{\mathrm{T}})\restriction D_1$ into $I(01)$, and C be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose f is a homeomorphism and $C \subseteq D_1$ and there exist points p_1, p_2 of $\mathbb{I}$ such that $p_1 < p_2$ and $f^{\circ}C = [p_1, p_2]$. Then there exist points s_1, s_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that C is an arc from s_1 to s_2.

(81) Let D be a simple closed curve and C be a non empty compact connected subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $C \subseteq D$. Then $C = D$ or there exist points p_1, p_2 of $\mathcal{E}^2_{\mathrm{T}}$ such that C is an arc from p_1 to p_2 or there exists a point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $C = \{p\}$.

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