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The Canonical Formulae

Andrzej Trybulec
University of Białystok

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The articles [23], [29], [11], [28], [14], [2], [27], [12], [30], [8], [5], [3], [20], [9], [6], [22], [7], [10], [1], [4], [15], [17], [18], [24], [25], [19], [16], [21], [13], and [26] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For every integer i holds i is even iff $i-1$ is odd.

(2) For every integer i holds i is odd iff $i-1$ is even.

(3) Let X be a trivial set and x be a set. Suppose $x \in X$. Let f be a function from X into X. Then x is a fixpoint of f.

Let A, B, C be sets. Note that every function from A into C^B is function yielding.

One can prove the following three propositions:

(4) For every function yielding function f holds $\mathrm{Sub_f}\,\mathrm{rng}\, f = \mathrm{rng}\, f$.

(5) For all sets A, B, x and for every function f such that $x \in A$ and $f \in B^A$ holds $f(x) \in B$.

(6) For all sets A, B, C such that if $C = \emptyset$, then $B = \emptyset$ or $A = \emptyset$ and for every function f from A into C^B holds $\mathrm{dom}_\kappa\, f(\kappa) = A \longmapsto B$.

Let us note that $\emptyset$ is function yielding.

In the sequel n is a natural number and p, q, r are elements of HP-WFF.

Next we state the proposition

(7) For every set x holds $\emptyset(x) = \emptyset$.

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Let A be a set and let B be a functional set. One can verify that every function from A into B is function yielding.

One can prove the following propositions:

(8) For every set X and for every subset A of X holds $[0 \longmapsto 1, 1 \longmapsto 0] \cdot \chi_{A,X} = \chi_{A^{\mathrm{c}},X}$.

(9) For every set X and for every subset A of X holds $[0 \longmapsto 1, 1 \longmapsto 0] \cdot \chi_{A^{\mathrm{c}},X} = \chi_{A,X}$.

(10) For all sets a, b, x, y, x', y' such that $a \neq b$ and $[a \longmapsto x, b \longmapsto y] = [a \longmapsto x', b \longmapsto y']$ holds $x = x'$ and $y = y'$.

(11) For all sets a, b, x, y, X, Y such that $a \neq b$ and $x \in X$ and $y \in Y$ holds $[a \longmapsto x, b \longmapsto y] \in \prod[a \longmapsto X, b \longmapsto Y]$.

(12) For every non empty set D and for every function f from 2 into D there exist elements d_1, d_2 of D such that $f = [0 \longmapsto d_1, 1 \longmapsto d_2]$.

(13) For all sets a, b, c, d such that $a \neq b$ holds $[a \longmapsto c, b \longmapsto d] \cdot [a \longmapsto b, b \longmapsto a] = [a \longmapsto d, b \longmapsto c]$.

(14) For all sets a, b, c, d and for every function f such that $a \neq b$ and $c \in \mathrm{dom}\, f$ and $d \in \mathrm{dom}\, f$ holds $f \cdot [a \longmapsto c, b \longmapsto d] = [a \longmapsto f(c), b \longmapsto f(d)]$.

2. The Cartesian Product of Functions and the Frege Function

Let f, g be one-to-one functions. Note that $[: f, g :]$ is one-to-one.

We now state a number of propositions:

(15) Let A, B be non empty sets, C, D be sets, f be a function from C into A, and g be a function from D into B. Then $\pi_1(A \times B) \cdot [: f, g :] = f \cdot \pi_1(C \times D)$.

(16) Let A, B be non empty sets, C, D be sets, f be a function from C into A, and g be a function from D into B. Then $\pi_2(A \times B) \cdot [: f, g :] = g \cdot \pi_2(C \times D)$.

(17) For every function g holds $\emptyset \looparrowleft g = \emptyset$.

(18) For every function yielding function f and for all functions g, h holds $f \looparrowleft g \cdot h = (f \cdot h) \looparrowleft (g \cdot h)$.

(19) Let C be a set, A be a non empty set, f be a function from A into $C^{(\emptyset\ \mathbf{qua}\ \mathrm{set})}$, and g be a function from A into $\emptyset$. Then $\mathrm{rng}(f \looparrowleft g) = \{\emptyset\}$.

(20) Let A, B, C be sets such that if $B = \emptyset$, then $A = \emptyset$. Let f be a function from A into C^B and g be a function from A into B. Then $\mathrm{rng}(f \looparrowleft g) \subseteq C$.

(21) For all sets A, B, C such that if $C = \emptyset$, then $B = \emptyset$ or $A = \emptyset$ and for every function f from A into C^B holds $\mathrm{dom}\, \mathrm{Frege}(f) = B^A$.

(22) $\mathrm{Frege}(\emptyset) = \{\emptyset\} \longmapsto \emptyset$.

(23) For all sets A, B, C such that if $C = \emptyset$, then $B = \emptyset$ or $A = \emptyset$ and for every function f from A into C^B holds $\mathrm{rng}\, \mathrm{Frege}(f) \subseteq C^A$.

(24) Let A, B, C be sets such that if $C = \emptyset$, then $B = \emptyset$ or $A = \emptyset$. Let f be a function from A into C^B. Then $\mathrm{Frege}(f)$ is a function from B^A into C^A.

3. About Permutations

Let A be a set. Observe that every permutation of A is one-to-one.

The following proposition is true

(25) For all sets A, B and for every permutation P of A and for every permutation Q of B holds $[\!: P, Q :\!]$ is permutation-like.

Let A, B be non empty sets, let P be a permutation of A, and let Q be a function from B into B. The functor $P \Rightarrow Q$ yielding a function from B^A into B^A is defined as follows:

(Def. 1) For every function f from A into B holds $(P \Rightarrow Q)(f) = Q \cdot f \cdot P^{-1}$.

Let A, B be non empty sets, let P be a permutation of A, and let Q be a permutation of B. Observe that $P \Rightarrow Q$ is permutation-like.

Next we state three propositions:

(26) Let A, B be non empty sets, P be a permutation of A, Q be a permutation of B, and f be a function from A into B. Then $(P \Rightarrow Q)^{-1}(f) = Q^{-1} \cdot f \cdot P$.

(27) For all non empty sets A, B and for every permutation P of A and for every permutation Q of B holds $(P \Rightarrow Q)^{-1} = P^{-1} \Rightarrow Q^{-1}$.

(28) Let A, B, C be non empty sets, f be a function from A into C^B, g be a function from A into B, P be a permutation of B, and Q be a permutation of C. Then $((P \Rightarrow Q) \cdot f) \looparrowleft (P \cdot g) = Q \cdot f \looparrowleft g$.

4. Set Valuations

A SetValuation is a non-empty many sorted set indexed by $\mathbb{N}$.

In the sequel V denotes a SetValuation.

Let us consider V. The functor $\mathrm{SetVal}\, V$ yielding a many sorted set indexed by HP-WFF is defined by the conditions (Def. 2).

(Def. 2)(i) $(\mathrm{SetVal}\, V)(\mathrm{VERUM}) = 1$,

(ii) for every n holds $(\mathrm{SetVal}\, V)(\mathrm{prop}\, n) = V(n)$, and

(iii) for all p, q holds $(\mathrm{SetVal}\, V)(p \wedge q) = [\!: (\mathrm{SetVal}\, V)(p), (\mathrm{SetVal}\, V)(q) :\!]$ and $(\mathrm{SetVal}\, V)(p \Rightarrow q) = (\mathrm{SetVal}\, V)(q)^{(\mathrm{SetVal}\, V)(p)}$.

Let us consider V, p. The functor $\mathrm{SetVal}(V, p)$ is defined as follows:

(Def. 3) $\mathrm{SetVal}(V, p) = (\mathrm{SetVal}\, V)(p)$.

Let us consider V, p. One can check that $\mathrm{SetVal}(V,p)$ is non empty.

Next we state four propositions:

(29) $\mathrm{SetVal}(V, \mathrm{VERUM}) = 1$.

(30) $\mathrm{SetVal}(V, \mathrm{prop}\, n) = V(n)$.

(31) $\mathrm{SetVal}(V, p \wedge q) = [\!:\mathrm{SetVal}(V,p), \mathrm{SetVal}(V,q):\!]$.

(32) $\mathrm{SetVal}(V, p \Rightarrow q) = (\mathrm{SetVal}(V,q))^{\mathrm{SetVal}(V,p)}$.

Let us consider V, p, q. Observe that $\mathrm{SetVal}(V, p \Rightarrow q)$ is functional.

Let us consider V, p, q, r. Note that every element of $\mathrm{SetVal}(V, p \Rightarrow (q \Rightarrow r))$ is function yielding.

Let us consider V, p, q, r. One can check that there exists a function from $\mathrm{SetVal}(V, p \Rightarrow q)$ into $\mathrm{SetVal}(V, p \Rightarrow r)$ which is function yielding and there exists an element of $\mathrm{SetVal}(V, p \Rightarrow (q \Rightarrow r))$ which is function yielding.

5. Permuting Set Valuations

Let us consider V. A function is called a permutation of V if:

(Def. 4) $\mathrm{dom}\,\mathrm{it} = \mathbb{N}$ and for every n holds $\mathrm{it}(n)$ is a permutation of $V(n)$.

In the sequel P is a permutation of V.

Let us consider V, P. The functor $\mathrm{Perm}\, P$ yielding a many sorted function from $\mathrm{SetVal}\, V$ into $\mathrm{SetVal}\, V$ is defined by the conditions (Def. 5).

(Def. 5)(i) $(\mathrm{Perm}\, P)(\mathrm{VERUM}) = \mathrm{id}_1$,

(ii) for every n holds $(\mathrm{Perm}\, P)(\mathrm{prop}\, n) = P(n)$, and

(iii) for all p, q there exists a permutation p' of $\mathrm{SetVal}(V,p)$ and there exists a permutation q' of $\mathrm{SetVal}(V,q)$ such that $p' = (\mathrm{Perm}\, P)(p)$ and $q' = (\mathrm{Perm}\, P)(q)$ and $(\mathrm{Perm}\, P)(p \wedge q) = [\!:p', q':\!]$ and $(\mathrm{Perm}\, P)(p \Rightarrow q) = p' \Rightarrow q'$.

Let us consider V, P, p. The functor $\mathrm{Perm}(P,p)$ yields a function from $\mathrm{SetVal}(V,p)$ into $\mathrm{SetVal}(V,p)$ and is defined by:

(Def. 6) $\mathrm{Perm}(P,p) = (\mathrm{Perm}\, P)(p)$.

Next we state four propositions:

(33) $\mathrm{Perm}(P, \mathrm{VERUM}) = \mathrm{id}_{\mathrm{SetVal}(V, \mathrm{VERUM})}$.

(34) $\mathrm{Perm}(P, \mathrm{prop}\, n) = P(n)$.

(35) $\mathrm{Perm}(P, p \wedge q) = [\!:\mathrm{Perm}(P,p), \mathrm{Perm}(P,q):\!]$.

(36) For every permutation p' of $\mathrm{SetVal}(V,p)$ and for every permutation q' of $\mathrm{SetVal}(V,q)$ such that $p' = \mathrm{Perm}(P,p)$ and $q' = \mathrm{Perm}(P,q)$ holds $\mathrm{Perm}(P, p \Rightarrow q) = p' \Rightarrow q'$.

Let us consider V, P, p. One can check that $\mathrm{Perm}(P,p)$ is permutation-like.

We now state four propositions:

(37) For every function g from $\mathrm{SetVal}(V,p)$ into $\mathrm{SetVal}(V,q)$ holds $(\mathrm{Perm}(P,p \Rightarrow q))(g) = \mathrm{Perm}(P,q) \cdot g \cdot (\mathrm{Perm}(P,p))^{-1}$.

(38) For every function g from $\mathrm{SetVal}(V,p)$ into $\mathrm{SetVal}(V,q)$ holds $(\mathrm{Perm}(P,p \Rightarrow q))^{-1}(g) = (\mathrm{Perm}(P,q))^{-1} \cdot g \cdot \mathrm{Perm}(P,p)$.

(39) For all functions f, g from $\mathrm{SetVal}(V,p)$ into $\mathrm{SetVal}(V,q)$ such that $f = (\mathrm{Perm}(P,p \Rightarrow q))(g)$ holds $\mathrm{Perm}(P,q) \cdot g = f \cdot \mathrm{Perm}(P,p)$.

(40) Let given V, P be a permutation of V, and x be a set. Suppose x is a fixpoint of $\mathrm{Perm}(P,p)$. Let f be a function. If f is a fixpoint of $\mathrm{Perm}(P,p \Rightarrow q)$, then $f(x)$ is a fixpoint of $\mathrm{Perm}(P,q)$.

6. Canonical Formulae

Let us consider p. We say that p is canonical if and only if:

(Def. 7) For every V there exists a set x such that for every permutation P of V holds x is a fixpoint of $\mathrm{Perm}(P,p)$.

Let us observe that VERUM is canonical.

Next we state several propositions:

(41) $p \Rightarrow (q \Rightarrow p)$ is canonical.

(42) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ is canonical.

(43) $p \wedge q \Rightarrow p$ is canonical.

(44) $p \wedge q \Rightarrow q$ is canonical.

(45) $p \Rightarrow (q \Rightarrow p \wedge q)$ is canonical.

(46) If p is canonical and $p \Rightarrow q$ is canonical, then q is canonical.

(47) If $p \in$ HP_TAUT, then p is canonical.

Let us observe that there exists an element of HP-WFF which is canonical.

7. Pseudo-Canonical Formulae

Let us consider p. We say that p is pseudo-canonical if and only if:

(Def. 8) For every V and for every permutation P of V holds there exists a set which is a fixpoint of $\mathrm{Perm}(P,p)$.

Let us observe that every element of HP-WFF which is canonical is also pseudo-canonical.

One can prove the following propositions:

(48) $p \Rightarrow (q \Rightarrow p)$ is pseudo-canonical.

(49) $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ is pseudo-canonical.

(50) $p \wedge q \Rightarrow p$ is pseudo-canonical.

(51) $p \wedge q \Rightarrow q$ is pseudo-canonical.

(52) $p \Rightarrow (q \Rightarrow p \wedge q)$ is pseudo-canonical.

(53) If p is pseudo-canonical and $p \Rightarrow q$ is pseudo-canonical, then q is pseudo-canonical.

(54) Let given p, q, given V, and P be a permutation of V. Suppose there exists a set which is a fixpoint of $\mathrm{Perm}(P, p)$ and there exists no set which is a fixpoint of $\mathrm{Perm}(P, q)$. Then $p \Rightarrow q$ is not pseudo-canonical.

(55) $((\mathrm{prop}\,0 \Rightarrow \mathrm{prop}\,1) \Rightarrow \mathrm{prop}\,0) \Rightarrow \mathrm{prop}\,0$ is not pseudo-canonical.

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