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The Concept of Fuzzy Relation and Basic Properties of its Operation

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Summary. This article introduces the fuzzy relation. This is the expansion of usual relation, and the value is given at the fuzzy value. At first, the definition of the fuzzy relation characterized by membership function is described. Next, the definitions of the zero relation and universe relation and basic operations of these relations are shown.

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The papers [8], [1], [5], [9], [3], [4], [6], [7], and [2] provide the terminology and notation for this paper.

1. Definition of Fuzzy Relation

In this paper C_1, C_2 are non empty sets.

Let us consider C_1, C_2. A partial function from $[: C_1, C_2 :]$ to $\mathbb{R}$ is said to be a Membership function of C_1, C_2 if:

(Def. 1) $\operatorname{dom} \mathrm{it} = [: C_1, C_2 :]$ and $\operatorname{rng} \mathrm{it} \subseteq [0, 1]$.

The following proposition is true

(1) $\chi_{[: C_1, C_2 :], [: C_1, C_2 :]}$ is a Membership function of C_1, C_2.

Let C_1, C_2 be non empty sets and let h be a Membership function of C_1, C_2. A set is called a fuzzy relation of C_1, C_2, h if:

(Def. 2) $\mathrm{It} = [: [: C_1, C_2 :], h^{\circ}[: C_1, C_2 :] :]$.

Let C_1, C_2 be non empty sets, let h, g be Membership functions of C_1, C_2, let A be a fuzzy relation of C_1, C_2, h, and let B be a fuzzy relation of C_1, C_2, g. The predicate $A = B$ is defined by:

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(Def. 3) For every element c of $[: C_1, C_2 :]$ holds $h(c) = g(c)$.

Let C_1, C_2 be non empty sets, let h, g be Membership functions of C_1, C_2, let A be a fuzzy relation of C_1, C_2, h, and let B be a fuzzy relation of C_1, C_2, g. The predicate $A \subseteq B$ is defined by:

(Def. 4) For every element c of $[: C_1, C_2 :]$ holds $h(c) \leqslant g(c)$.

For simplicity, we adopt the following rules: f, g, h, h_1 denote Membership functions of C_1, C_2, A denotes a fuzzy relation of C_1, C_2, f, B denotes a fuzzy relation of C_1, C_2, g, D denotes a fuzzy relation of C_1, C_2, h, and D_1 denotes a fuzzy relation of C_1, C_2, h_1.

The following three propositions are true:

(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

(3) $A \subseteq A$.

(4) If $A \subseteq B$ and $B \subseteq D$, then $A \subseteq D$.

2. Intersection, Union and Complement

Let C_1, C_2 be non empty sets and let h, g be Membership functions of C_1, C_2. The functor $\min(h, g)$ yielding a Membership function of C_1, C_2 is defined as follows:

(Def. 5) For every element c of $[: C_1, C_2 :]$ holds $(\min(h, g))(c) = \min(h(c), g(c))$.

Let C_1, C_2 be non empty sets and let h, g be Membership functions of C_1, C_2. The functor $\max(h, g)$ yields a Membership function of C_1, C_2 and is defined by:

(Def. 6) For every element c of $[: C_1, C_2 :]$ holds $(\max(h, g))(c) = \max(h(c), g(c))$.

Let C_1, C_2 be non empty sets and let h be a Membership function of C_1, C_2. The functor 1-minus h yields a Membership function of C_1, C_2 and is defined as follows:

(Def. 7) For every element c of $[: C_1, C_2 :]$ holds $(\text{1-minus}\, h)(c) = 1 - h(c)$.

Let C_1, C_2 be non empty sets, let h, g be Membership functions of C_1, C_2, let A be a fuzzy relation of C_1, C_2, h, and let B be a fuzzy relation of C_1, C_2, g. The functor $A \cap B$ yields a fuzzy relation of C_1, C_2, $\min(h, g)$ and is defined as follows:

(Def. 8) $A \cap B = [: [: C_1, C_2 :], (\min(h, g))^{\circ}[: C_1, C_2 :] :]$.

Let C_1, C_2 be non empty sets, let h, g be Membership functions of C_1, C_2, let A be a fuzzy relation of C_1, C_2, h, and let B be a fuzzy relation of C_1, C_2, g. The functor $A \cup B$ yielding a fuzzy relation of C_1, C_2, $\max(h, g)$ is defined by:

(Def. 9) $A \cup B = [: [: C_1, C_2 :], (\max(h, g))^{\circ}[: C_1, C_2 :] :]$.

Let C_1, C_2 be non empty sets, let h be a Membership function of C_1, C_2, and let A be a fuzzy relation of C_1, C_2, h. The functor $A^{\rm c}$ yielding a fuzzy relation of C_1, C_2, 1-minus h is defined as follows:

(Def. 10) $A^{\rm c} = [\![[\![C_1, C_2]\!], (\text{1-minus}\, h)^\circ [\![C_1, C_2]\!]]\!]$.

The following propositions are true:

(5) For every element x of $[\![C_1, C_2]\!]$ holds $\min(h(x), g(x)) = (\min(h, g))(x)$ and $\max(h(x), g(x)) = (\max(h, g))(x)$.

(6) $\max(h, h) = h$ and $\min(h, h) = h$ and $\max(h, h) = \min(h, h)$ and $\min(f, g) = \min(g, f)$ and $\max(f, g) = \max(g, f)$.

(7) $f = g$ iff $A = B$.

(8) $A \cap A = A$ and $A \cup A = A$.

(9) $A \cap B = B \cap A$ and $A \cup B = B \cup A$.

(10) $\max(\max(f, g), h) = \max(f, \max(g, h))$ and $\min(\min(f, g), h) = \min(f, \min(g, h))$.

(11) $(A \cup B) \cup D = A \cup (B \cup D)$.

(12) $(A \cap B) \cap D = A \cap (B \cap D)$.

(13) $\max(f, \min(f, g)) = f$ and $\min(f, \max(f, g)) = f$.

(14) $A \cup A \cap B = A$ and $A \cap (A \cup B) = A$.

(15) $\min(f, \max(g, h)) = \max(\min(f, g), \min(f, h))$ and $\max(f, \min(g, h)) = \min(\max(f, g), \max(f, h))$.

(16) $A \cup B \cap D = (A \cup B) \cap (A \cup D)$ and $A \cap (B \cup D) = A \cap B \cup A \cap D$.

(17) 1-minus 1-minus $h = h$.

(18) $(A^{\rm c})^{\rm c} = A$.

(19) 1-minus $\max(f, g) = \min(\text{1-minus}\, f, \text{1-minus}\, g)$ and 1-minus $\min(f, g) = \max(\text{1-minus}\, f, \text{1-minus}\, g)$.

(20) $(A \cup B)^{\rm c} = A^{\rm c} \cap B^{\rm c}$ and $(A \cap B)^{\rm c} = A^{\rm c} \cup B^{\rm c}$.

(21) $A \subseteq A \cup B$.

(22) If $A \subseteq D$ and $B \subseteq D$, then $A \cup B \subseteq D$.

(23) If $A \subseteq B$, then $A \cup D \subseteq B \cup D$.

(24) If $A \subseteq B$ and $D \subseteq D_1$, then $A \cup D \subseteq B \cup D_1$.

(25) If $A \subseteq B$, then $A \cup B = B$.

(26) $A \cap B \subseteq A$.

(27) $A \cap B \subseteq A \cup B$.

(28) If $D \subseteq A$ and $D \subseteq B$, then $D \subseteq A \cap B$.

(29) For all elements a, b, c, d of $\mathbb{R}$ such that $a \leqslant b$ and $c \leqslant d$ holds $\min(a, c) \leqslant \min(b, d)$.

(30) For all elements a, b, c of $\mathbb{R}$ such that $a \leqslant b$ holds $\min(a, c) \leqslant \min(b, c)$.

(31) If $A \subseteq B$, then $A \cap D \subseteq B \cap D$.

(32) If $A \subseteq B$ and $D \subseteq D_1$, then $A \cap D \subseteq B \cap D_1$.

(33) If $A \subseteq B$, then $A \cap B = A$.

(34) If $A \cap B \cup A \cap D = A$, then $A \subseteq B \cup D$.

(35) $A = B \cup D$ iff $B \subseteq A$ and $D \subseteq A$ and for all h_1, D_1 such that $B \subseteq D_1$ and $D \subseteq D_1$ holds $A \subseteq D_1$.

(36) $A = B \cap D$ iff $A \subseteq B$ and $A \subseteq D$ and for all h_1, D_1 such that $D_1 \subseteq B$ and $D_1 \subseteq D$ holds $D_1 \subseteq A$.

(37) $A \subseteq B$ iff $B^{\rm c} \subseteq A^{\rm c}$.

(38) If $A \subseteq B^{\rm c}$, then $B \subseteq A^{\rm c}$.

(39) If $A^{\rm c} \subseteq B$, then $B^{\rm c} \subseteq A$.

(40) $(A \cup B)^{\rm c} \subseteq A^{\rm c}$ and $(A \cup B)^{\rm c} \subseteq B^{\rm c}$.

(41) $A^{\rm c} \subseteq (A \cap B)^{\rm c}$ and $B^{\rm c} \subseteq (A \cap B)^{\rm c}$.

3. Exclusive Sum

Let C_1, C_2 be non empty sets, let h, g be Membership functions of C_1, C_2, let A be a fuzzy relation of C_1, C_2, h, and let B be a fuzzy relation of C_1, C_2, g. The functor $A \dot{-} B$ yields a fuzzy relation of C_1, C_2, $\max(\min(h, 1\text{-minus}\, g),$ $\min(1\text{-minus}\, h, g))$ and is defined by:

(Def. 11) $A \dot{-} B = [\![[\![C_1, C_2]\!], (\max(\min(h, 1\text{-minus}\, g), \min(1\text{-minus}\, h, g)))^{\circ}[\![C_1, C_2]\!]]\!]$.

The following propositions are true:

(42) $A \dot{-} B = A \cap B^{\rm c} \cup A^{\rm c} \cap B$.

(43) $A \dot{-} B = B \dot{-} A$.

4. Zero Relation and Universe Relation

Let C_1, C_2 be non empty sets. A set is called a zero relation of C_1, C_2 if:

(Def. 12) It $= [\![[\![C_1, C_2]\!], (\chi_{\emptyset, [\![C_1, C_2]\!]})^{\circ}[\![C_1, C_2]\!]]\!]$.

Let C_1, C_2 be non empty sets. A set is called a universe relation of C_1, C_2 if:

(Def. 13) It $= [\![[\![C_1, C_2]\!], (\chi_{[\![C_1, C_2]\!], [\![C_1, C_2]\!]})^{\circ}[\![C_1, C_2]\!]]\!]$.

In the sequel X is a universe relation of C_1, C_2 and O is a zero relation of C_1, C_2.

The following proposition is true

(44) $\chi_{\emptyset, [\![C_1, C_2]\!]}$ is a Membership function of C_1, C_2.

Let C_1, C_2 be non empty sets. The functor $\mathrm{Zmf}(C_1, C_2)$ yielding a Membership function of C_1, C_2 is defined as follows:

(Def. 14) $\mathrm{Zmf}(C_1, C_2) = \chi_{\emptyset, [: C_1, C_2 :]}$.

Let C_1, C_2 be non empty sets. The functor $\mathrm{Umf}(C_1, C_2)$ yields a Membership function of C_1, C_2 and is defined as follows:

(Def. 15) $\mathrm{Umf}(C_1, C_2) = \chi_{[: C_1, C_2 :], [: C_1, C_2 :]}$.

Next we state four propositions:

(45) Let h be a Membership function of C_1, C_2. If $h = \chi_{[: C_1, C_2 :], [: C_1, C_2 :]}$, then $[: [: C_1, C_2 :], (\chi_{[: C_1, C_2 :], [: C_1, C_2 :]})^{\circ}[: C_1, C_2 :] :]$ is a fuzzy relation of C_1, C_2, h.

(46) For every Membership function h of C_1, C_2 such that $h = \chi_{\emptyset, [: C_1, C_2 :]}$ holds $[: [: C_1, C_2 :], (\chi_{\emptyset, [: C_1, C_2 :]})^{\circ}[: C_1, C_2 :] :]$ is a fuzzy relation of C_1, C_2, h.

(47) O is a fuzzy relation of C_1, C_2, $\mathrm{Zmf}(C_1, C_2)$.

(48) X is a fuzzy relation of C_1, C_2, $\mathrm{Umf}(C_1, C_2)$.

Let C_1, C_2 be non empty sets. We see that the zero relation of C_1, C_2 is a fuzzy relation of C_1, C_2, $\mathrm{Zmf}(C_1, C_2)$.

Let C_1, C_2 be non empty sets. We see that the universe relation of C_1, C_2 is a fuzzy relation of C_1, C_2, $\mathrm{Umf}(C_1, C_2)$.

In the sequel X denotes a universe relation of C_1, C_2 and O denotes a zero relation of C_1, C_2.

Next we state a number of propositions:

(49) Let a, b be elements of $\mathbb{R}$ and f be a partial function from $[: C_1, C_2 :]$ to $\mathbb{R}$. Suppose $\mathrm{rng}\, f \subseteq [a, b]$ and $\mathrm{dom}\, f \neq \emptyset$ and $a \leqslant b$. Let x be an element of $[: C_1, C_2 :]$. If $x \in \mathrm{dom}\, f$, then $a \leqslant f(x)$ and $f(x) \leqslant b$.

(50) $O \subseteq A$.

(51) $A \subseteq X$.

(52) For every element x of $[: C_1, C_2 :]$ and for every Membership function h of C_1, C_2 holds $(\mathrm{Zmf}(C_1, C_2))(x) \leqslant h(x)$ and $h(x) \leqslant (\mathrm{Umf}(C_1, C_2))(x)$.

(53) $\max(f, \mathrm{Umf}(C_1, C_2)) = \mathrm{Umf}(C_1, C_2)$ and $\min(f, \mathrm{Umf}(C_1, C_2)) = f$ and $\max(f, \mathrm{Zmf}(C_1, C_2)) = f$ and $\min(f, \mathrm{Zmf}(C_1, C_2)) = \mathrm{Zmf}(C_1, C_2)$.

(54) $A \cup X = X$ and $A \cap X = A$.

(55) $A \cup O = A$ and $A \cap O = O$.

(56) If $A \subseteq B$ and $A \subseteq D$ and $B \cap D = O$, then $A = O$.

(57) If $A \subseteq B$ and $B \cap D = O$, then $A \cap D = O$.

(58) If $A \subseteq O$, then $A = O$.

(59) $A \cup B = O$ iff $A = O$ and $B = O$.

(60) If $A \subseteq B \cup D$ and $A \cap D = O$, then $A \subseteq B$.

(61) $\text{1-minus}\, \mathrm{Zmf}(C_1, C_2) = \mathrm{Umf}(C_1, C_2)$ and $\text{1-minus}\, \mathrm{Umf}(C_1, C_2) = \mathrm{Zmf}(C_1, C_2)$.

(62) $O^{\rm c} = X$ and $X^{\rm c} = O$.

(63) $A \dot{-} O = A$ and $O \dot{-} A = A$.

(64) $A \dot{-} X = A^{\rm c}$ and $X \dot{-} A = A^{\rm c}$.

(65) For every element c of $[: C_1, C_2 :]$ such that $f(c) \leqslant h(c)$ holds $(\max(f, \min(g, h)))(c) = (\min(\max(f, g), h))(c)$.

(66) If $A \subseteq D$, then $A \cup B \cap D = (A \cup B) \cap D$.

Let C_1, C_2 be non empty sets, let f, g be Membership functions of C_1, C_2, let A be a fuzzy relation of C_1, C_2, f, and let B be a fuzzy relation of C_1, C_2, g. The functor $A \setminus B$ yielding a fuzzy relation of C_1, C_2, $\min(f, 1\text{-minus}\, g)$ is defined by:

(Def. 16) $A \setminus B = [: [: C_1, C_2 :], (\min(f, 1\text{-minus}\, g))^{\circ}[: C_1, C_2 :] :]$.

One can prove the following propositions:

(67) $A \setminus B = A \cap B^{\rm c}$.

(68) $1\text{-minus} \min(f, 1\text{-minus}\, g) = \max(1\text{-minus}\, f, g)$.

(69) $(A \setminus B)^{\rm c} = A^{\rm c} \cup B$.

(70) For every element c of $[: C_1, C_2 :]$ such that $f(c) \leqslant g(c)$ holds $(\min(f, 1\text{-minus}\, h))(c) \leqslant (\min(g, 1\text{-minus}\, h))(c)$.

(71) If $A \subseteq B$, then $A \setminus D \subseteq B \setminus D$.

(72) For every element c of $[: C_1, C_2 :]$ such that $f(c) \leqslant g(c)$ holds $(\min(h, 1\text{-minus}\, g))(c) \leqslant (\min(h, 1\text{-minus}\, f))(c)$.

(73) If $A \subseteq B$, then $D \setminus B \subseteq D \setminus A$.

(74) For every element c of $[: C_1, C_2 :]$ such that $f(c) \leqslant g(c)$ and $h(c) \leqslant h_1(c)$ holds $(\min(f, 1\text{-minus}\, h_1))(c) \leqslant (\min(g, 1\text{-minus}\, h))(c)$.

(75) If $A \subseteq B$ and $D \subseteq D_1$, then $A \setminus D_1 \subseteq B \setminus D$.

(76) For every element c of $[: C_1, C_2 :]$ holds $(\min(f, 1\text{-minus}\, g))(c) \leqslant f(c)$.

(77) $A \setminus B \subseteq A$.

(78) For every element c of $[: C_1, C_2 :]$ holds $(\min(f, 1\text{-minus}\, g))(c) \leqslant (\max(\min(f, 1\text{-minus}\, g), \min(1\text{-minus}\, f, g)))(c)$.

(79) $A \setminus B \subseteq A \dot{-} B$.

(80) $A \setminus O = A$.

(81) $O \setminus A = O$.

(82) For every element c of $[: C_1, C_2 :]$ holds $(\min(f, 1\text{-minus}\, g))(c) \leqslant (\min(f, 1\text{-minus} \min(f, g)))(c)$.

(83) $A \setminus B \subseteq A \setminus A \cap B$.

(84) For every element c of $[: C_1, C_2 :]$ holds $(\max(\min(f, g), \min(f, 1\text{-minus}\, g)))(c) \leqslant f(c)$.

(85) For every element c of $[: C_1, C_2 :]$ holds $(\max(f, \min(g, 1\text{-minus}\, f)))(c) \leqslant (\max(f, g))(c)$.

(86) $A \cup (B \setminus A) \subseteq A \cup B$.

(87) $A \cap B \cup (A \setminus B) \subseteq A$.

(88) $\min(f, \text{1-minus} \min(g, \text{1-minus}\, h)) = \max(\min(f, \text{1-minus}\, g), \min(f, h))$.

(89) $A \setminus (B \setminus D) = (A \setminus B) \cup A \cap D$.

(90) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\min(f, g))(c) \leqslant (\min(f, \text{1-minus} \min(f, \text{1-minus}\, g)))(c)$.

(91) $A \cap B \subseteq A \setminus (A \setminus B)$.

(92) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\min(f, \text{1-minus}\, g))(c) \leqslant (\min(\max(f, g), \text{1-minus}\, g))(c)$.

(93) $A \setminus B \subseteq (A \cup B) \setminus B$.

(94) $\min(f, \text{1-minus} \max(g, h)) = \min(\min(f, \text{1-minus}\, g), \min(f, \text{1-minus}\, h))$.

(95) $A \setminus (B \cup D) = (A \setminus B) \cap (A \setminus D)$.

(96) $\min(f, \text{1-minus} \min(g, h)) = \max(\min(f, \text{1-minus}\, g), \min(f, \text{1-minus}\, h))$.

(97) $A \setminus B \cap D = (A \setminus B) \cup (A \setminus D)$.

(98) $\min(\min(f, \text{1-minus}\, g), \text{1-minus}\, h) = \min(f, \text{1-minus} \max(g, h))$.

(99) $A \setminus B \setminus D = A \setminus (B \cup D)$.

(100) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\min(\max(f, g), \text{1-minus} \min(f, g)))(c) \geqslant (\max(\min(f, \text{1-minus}\, g), \min(g, \text{1-minus}\, f)))(c)$.

(101) $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus A \cap B$.

(102) $\min(\max(f, g), \text{1-minus}\, h) = \max(\min(f, \text{1-minus}\, h), \min(g, \text{1-minus}\, h))$.

(103) $(A \cup B) \setminus D = (A \setminus D) \cup (B \setminus D)$.

(104) For every element c of $[\!: C_1, C_2 :\!]$ such that $(\min(f, \text{1-minus}\, g))(c) \leqslant h(c)$ and $(\min(g, \text{1-minus}\, f))(c) \leqslant h(c)$ holds $(\max(\min(f, \text{1-minus}\, g), \min(\text{1-minus}\, f, g)))(c) \leqslant h(c)$.

(105) If $A \setminus B \subseteq D$ and $B \setminus A \subseteq D$, then $A \dot{-} B \subseteq D$.

(106) $A \cap (B \setminus D) = A \cap B \setminus D$.

(107) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\min(f, \min(g, \text{1-minus}\, h)))(c) \leqslant (\min(\min(f, g), \text{1-minus} \min(f, h)))(c)$.

(108) $A \cap (B \setminus D) \subseteq A \cap B \setminus A \cap D$.

(109) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\min(\max(f, g), \text{1-minus} \min(f, g)))(c) \geqslant (\max(\min(f, \text{1-minus}\, g), \min(\text{1-minus}\, f, g)))(c)$.

(110) $A \dot{-} B \subseteq (A \cup B) \setminus A \cap B$.

(111) For every element c of $[\!: C_1, C_2 :\!]$ holds $(\max(\min(f, g), \text{1-minus} \max(f, g)))(c) \leqslant (\text{1-minus} \max(\min(f, \text{1-minus}\, g), \min(\text{1-minus}\, f, g)))(c)$.

(112) $A \cap B \cup (A \cup B)^{\mathrm{c}} \subseteq (A \dot{-} B)^{\mathrm{c}}$.

(113) $\min(\max(\min(f, \text{1-minus}\, g), \min(\text{1-minus}\, f, g)), \text{1-minus}\, h) = \max(\min(f, \text{1-minus} \max(g, h)), \min(g, \text{1-minus} \max(f, h)))$.

(114) $(A \dot{-} B) \setminus D = (A \setminus (B \cup D)) \cup (B \setminus (A \cup D))$.

(115) For every element c of $[: C_1, C_2 :]$ holds $(\min(f, 1\text{-minus} \max(\min(g, 1\text{-minus} h), \min(1\text{-minus} g, h))))(c) \geqslant (\max(\min(f, 1\text{-minus} \max(g, h)), \min(\min(f, g), h)))(c)$.

(116) $(A \setminus (B \cup D)) \cup A \cap B \cap D \subseteq A \setminus (B \dot{-} D)$.

(117) For every element c of $[: C_1, C_2 :]$ such that $f(c) \leqslant g(c)$ holds $g(c) \geqslant (\max(f, \min(g, 1\text{-minus} f)))(c)$.

(118) If $A \subseteq B$, then $A \cup (B \setminus A) \subseteq B$.

(119) For every element c of $[: C_1, C_2 :]$ holds $(\max(f, g))(c) \geqslant (\max(\max(\min(f, 1\text{-minus} g), \min(1\text{-minus} f, g)), \min(f, g)))(c)$.

(120) $(A \dot{-} B) \cup A \cap B \subseteq A \cup B$.

(121) If $\min(f, 1\text{-minus} g) = \text{Zmf}(C_1, C_2)$, then for every element c of $[: C_1, C_2 :]$ holds $f(c) \leqslant g(c)$.

(122) If $A \setminus B = O$, then $A \subseteq B$.

(123) If $\min(f, g) = \text{Zmf}(C_1, C_2)$, then $\min(f, 1\text{-minus} g) = f$.

(124) If $A \cap B = O$, then $A \setminus B = A$.

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