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The Characterization of the Continuity of Topologies[1]

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Summary. Formalization of [14, pp. 128–130], chapter II, section 4 (4.10, 4.11).

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The terminology and notation used here are introduced in the following articles: [27], [23], [13], [10], [9], [21], [1], [30], [28], [24], [32], [22], [25], [31], [26], [12], [34], [29], [17], [15], [20], [6], [8], [3], [4], [33], [19], [7], [2], [16], [18], [5], and [11].

1. Preliminaries

The following propositions are true:

(1) Let S, T be non empty relational structures and f be a map from S into T. Suppose f is one-to-one and onto. Then $f \cdot f^{-1} = \mathrm{id}_T$ and $f^{-1} \cdot f = \mathrm{id}_S$ and f^{-1} is one-to-one and onto.

(2) Let X, Y be non empty sets, Z be a non empty relational structure, S be a non empty relational substructure of $Z^{[:X,Y:]}$, T be a non empty relational substructure of $(Z^Y)^X$, and f be a map from S into T. If f is currying, one-to-one, and onto, then f^{-1} is uncurrying.

(3) Let X, Y be non empty sets, Z be a non empty relational structure, S be a non empty relational substructure of $Z^{[:X,Y:]}$, T be a non empty relational substructure of $(Z^Y)^X$, and f be a map from T into S. If f is uncurrying, one-to-one, and onto, then f^{-1} is currying.

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(4) Let X, Y be non empty sets, Z be a non empty poset, S be a non empty full relational substructure of $Z^{[: X, Y :]}$, T be a non empty full relational substructure of $(Z^Y)^X$, and f be a map from S into T. If f is currying, one-to-one, and onto, then f is isomorphic.

(5) Let X, Y be non empty sets, Z be a non empty poset, T be a non empty full relational substructure of $Z^{[: X, Y :]}$, S be a non empty full relational substructure of $(Z^Y)^X$, and f be a map from S into T. If f is uncurrying, one-to-one, and onto, then f is isomorphic.

(6) Let S_1, S_2, T_1, T_2 be relational structures. Suppose that

(i) the relational structure of S_1 = the relational structure of S_2, and

(ii) the relational structure of T_1 = the relational structure of T_2.

Let f be a map from S_1 into T_1. Suppose f is isomorphic. Let g be a map from S_2 into T_2. If $g = f$, then g is isomorphic.

(7) Let R, S, T be relational structures and f be a map from R into S. Suppose f is isomorphic. Let g be a map from S into T. Suppose g is isomorphic. Let h be a map from R into T. If $h = g \cdot f$, then h is isomorphic.

(8) Let T be an up-complete Scott non empty top-poset and S be a subset of T. Then S is closed if and only if S is directly closed and lower.

(9) Let S, T be up-complete Scott non empty top-posets and f be a map from S into T. If f is directed-sups-preserving, then f is continuous.

(10) Let X, Y, X_1, Y_1 be topological spaces. Suppose that

(i) the topological structure of X = the topological structure of X_1, and

(ii) the topological structure of Y = the topological structure of Y_1.

Then $[: X, Y :] = [: X_1, Y_1 :]$.

(11) Let X be a non empty topological space, L be a Scott up-complete non empty top-poset, and F be a non empty directed subset of $[X \to L]$. Then $\bigsqcup_{(L^{\text{the carrier of } X})} F$ is a continuous map from X into L.

(12) Let X be a non empty topological space and L be a Scott up-complete non empty top-poset. Then $[X \to L]$ is a directed-sups-inheriting relational substructure of $L^{\text{the carrier of } X}$.

(13) Let S_1, S_2 be topological structures. Suppose the topological structure of S_1 = the topological structure of S_2. Let T_1, T_2 be non empty FR-structures. If the FR-structure of T_1 = the FR-structure of T_2, then $[S_1 \to T_1] = [S_2 \to T_2]$.

One can check that every complete continuous top-lattice which is Scott is also injective and T_0.

One can check that there exists a top-lattice which is Scott, continuous, and complete.

Let X be a non empty topological space and let L be a Scott up-complete non empty top-poset. Note that $[X \to L]$ is up-complete.

One can prove the following propositions:

(14) Let I be a non empty set and J be a poset-yielding nonempty many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is up-complete. Then I-prod$_{\rm POS}$ J is up-complete.

(15) Let I be a non empty set and J be a poset-yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is up-complete and lower-bounded. Let x, y be elements of $\prod J$. Then $x \ll y$ if and only if the following conditions are satisfied:

(i) for every element i of I holds $x(i) \ll y(i)$, and

(ii) there exists a finite subset K of I such that for every element i of I such that $i \notin K$ holds $x(i) = \bot_{J(i)}$.

Let X be a set and let L be a lower-bounded non empty reflexive antisymmetric relational structure. Observe that L^X is lower-bounded.

Let X be a non empty topological space and let L be a lower-bounded non empty top-poset. Note that $[X \to L]$ is lower-bounded.

Let L be an up-complete non empty poset. Note that every topological augmentation of L is up-complete and every topological augmentation of L which is Scott is also correct.

The following proposition is true

(16) Let S be an up-complete antisymmetric non empty reflexive relational structure and T be a non empty reflexive relational structure. Suppose the relational structure of S = the relational structure of T. Let A be a subset of S and C be a subset of T. If $A = C$ and A is inaccessible, then C is inaccessible.

Let L be an up-complete non empty poset. Observe that there exists a topological augmentation of L which is strict and Scott.

We now state two propositions:

(17) Let L be an up-complete non empty poset and S_1, S_2 be Scott topological augmentations of L. Then the topology of S_1 = the topology of S_2.

(18) Let S_1, S_2 be up-complete antisymmetric non empty reflexive FR-structures. Suppose the FR-structure of S_1 = the FR-structure of S_2 and S_1 is Scott. Then S_2 is Scott.

Let L be an up-complete non empty poset.

(Def. 1) ΣL is a strict Scott topological augmentation of L.

We now state two propositions:

(19) For every Scott up-complete non empty top-poset S holds ΣS = the FR-structure of S.

(20) Let L_1, L_2 be up-complete non empty posets. Suppose the relational structure of L_1 = the relational structure of L_2. Then $\Sigma L_1 = \Sigma L_2$.

Let S, T be up-complete non empty posets and let f be a map from S into T. The functor Σf yielding a map from ΣS into ΣT is defined as follows:

(Def. 2) $\Sigma f = f$.

Let S, T be up-complete non empty posets and let f be a directed-sups-preserving map from S into T. Observe that Σf is continuous.

One can prove the following propositions:

(21) Let S, T be up-complete non empty posets and f be a map from S into T. Then f is isomorphic if and only if Σf is isomorphic.

(22) For every non empty topological space X and for every Scott complete top-lattice S holds $[X \to S] = [X \to S]$.

Let X, Y be non empty topological spaces. The functor $\Theta(X,Y)$ yielding a map from $\langle$the topology of $[\!: X, Y :\!]$, $\subseteq\rangle$ into $[X \to \Sigma\langle$the topology of Y, $\subseteq\rangle]$ is defined as follows:

(Def. 3) For every open subset W of $[\!: X, Y :\!]$ holds $(\Theta(X,Y))(W) = \Theta_{\text{the carrier of } X}(W)$.

2. Some Natural Isomorphisms

Let X be a non empty topological space. The functor $\alpha(X)$ yielding a map from $[X \to$ the Sierpiński space$]$ into $\langle$the topology of X, $\subseteq\rangle$ is defined as follows:

(Def. 4) For every continuous map g from X into the Sierpiński space holds $(\alpha(X))(g) = g^{-1}(\{1\})$.

One can prove the following proposition

(23) For every non empty topological space X and for every open subset V of X holds $(\alpha(X))^{-1}(V) = \chi_{V,\text{the carrier of } X}$.

Let X be a non empty topological space. Note that $\alpha(X)$ is isomorphic.

Let X be a non empty topological space. One can verify that $(\alpha(X))^{-1}$ is isomorphic.

Let S be an injective T_0-space. One can verify that ΩS is Scott.

Let X be a non empty topological space. One can check that $[X \to$ the Sierpiński space$]$ is complete.

Next we state the proposition

(24) Ω(the Sierpiński space) $= \Sigma 2^1_\subseteq$.

Let M be a non empty set and let S be an injective T_0-space. One can verify that $M\text{-prod}_{\text{TOP}}(M \longmapsto S)$ is injective.

The following two propositions are true:

(25) For every non empty set M and for every complete continuous lattice L holds $\Omega(M\text{-prod}_{\text{TOP}}(M \longmapsto \Sigma L)) = \Sigma M\text{-prod}_{\text{POS}}(M \longmapsto L)$.

(26) For every non empty set M and for every injective T_0-space T holds $\Omega(M\text{-prod}_{\rm TOP}(M \longmapsto T)) = \Sigma M\text{-prod}_{\rm POS}(M \longmapsto \Omega T)$.

Let M be a non empty set and let X, Y be non empty topological spaces. The functor commute(X, M, Y) yielding a map from $[X \to M\text{-prod}_{\rm TOP}(M \longmapsto Y)]$ into $([X \to Y])^M$ is defined by:

(Def. 5) For every continuous map f from X into $M\text{-prod}_{\rm TOP}(M \longmapsto Y)$ holds $(\text{commute}(X, M, Y))(f) = \text{commute}(f)$.

Let M be a non empty set and let X, Y be non empty topological spaces. Note that commute(X, M, Y) is one-to-one and onto.

Let M be a non empty set and let X be a non empty topological space. Note that commute$(X, M,$ the Sierpiński space$)$ is isomorphic.

Next we state the proposition

(27) Let X, Y be non empty topological spaces, S be a Scott topological augmentation of $\langle$the topology of Y, $\subseteq\rangle$, and f_1, f_2 be elements of $[X \to S]$. If $f_1 \leqslant f_2$, then $G_{f_1} \subseteq G_{f_2}$.

3. The Poset of Open Sets

The following propositions are true:

(28) Let Y be a T_0-space. Then the following statements are equivalent

(i) for every non empty topological space X and for every Scott continuous complete top-lattice L and for every Scott topological augmentation T of $[Y \to L]$ there exists a map f from $[X \to T]$ into $[[\!: X, Y :\!] \to L]$ and there exists a map g from $[[\!: X, Y :\!] \to L]$ into $[X \to T]$ such that f is uncurrying, one-to-one, and onto and g is currying, one-to-one, and onto,

(ii) for every non empty topological space X and for every Scott continuous complete top-lattice L and for every Scott topological augmentation T of $[Y \to L]$ there exists a map f from $[X \to T]$ into $[[\!: X, Y :\!] \to L]$ and there exists a map g from $[[\!: X, Y :\!] \to L]$ into $[X \to T]$ such that f is uncurrying and isomorphic and g is currying and isomorphic.

(29) Let Y be a T_0-space. Then $\langle$the topology of Y, $\subseteq\rangle$ is continuous if and only if for every non empty topological space X holds $\Theta(X, Y)$ is isomorphic.

(30) Let Y be a T_0-space. Then $\langle$the topology of Y, $\subseteq\rangle$ is continuous if and only if for every non empty topological space X and for every continuous map f from X into $\Sigma\langle$the topology of Y, $\subseteq\rangle$ holds G_f is an open subset of $[\!: X, Y :\!]$.

(31) Let Y be a T_0-space. Then $\langle$the topology of Y, $\subseteq\rangle$ is continuous if and only if $\{\langle W, y\rangle; W$ ranges over open subsets of Y, y ranges over elements of Y: $y \in W\}$ is an open subset of $[\!: \Sigma\langle$the topology of Y, $\subseteq\rangle, Y :\!]$.

(32) Let Y be a T_0-space. Then $\langle$the topology of Y, $\subseteq\rangle$ is continuous if and only if for every element y of Y and for every open neighbourhood V of y there exists an open subset H of $\Sigma\langle$the topology of Y, $\subseteq\rangle$ such that $V \in H$ and $\bigcap H$ is a neighbourhood of y.

4. The Poset of Scott Open Sets

One can prove the following propositions:

(33) Let R_1, R_2, R_3 be non empty relational structures and f_1 be a map from R_1 into R_3. Suppose f_1 is isomorphic. Let f_2 be a map from R_2 into R_3. Suppose $f_2 = f_1$ and f_2 is isomorphic. Then the relational structure of $R_1 =$ the relational structure of R_2.

(34) Let L be a complete lattice. Then $\langle \sigma(L), \subseteq \rangle$ is continuous if and only if for every complete lattice S holds $\sigma([\!: S, L :\!]) =$ the topology of $[\!: \Sigma S, \Sigma L :\!]$.

(35) Let L be a complete lattice. Then the following statements are equivalent

(i) for every complete lattice S holds $\sigma([\!: S, L :\!]) =$ the topology of $[\!: \Sigma S, \Sigma L :\!]$,

(ii) for every complete lattice S holds the topological structure of $\Sigma[\!: S, L :\!] = [\!: \Sigma S, \Sigma L :\!]$.

(36) Let L be a complete lattice. Then for every complete lattice S holds $\sigma([\!: S, L :\!]) =$ the topology of $[\!: \Sigma S, \Sigma L :\!]$ if and only if for every complete lattice S holds $\Sigma[\!: S, L :\!] = \Omega[\!: \Sigma S, \Sigma L :\!]$.

(37) Let L be a complete lattice. Then $\langle \sigma(L), \subseteq \rangle$ is continuous if and only if for every complete lattice S holds $\Sigma[\!: S, L :\!] = \Omega[\!: \Sigma S, \Sigma L :\!]$.

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