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# Lim-Inf Convergence[1]

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**Summary.** This work continues the formalization of [7]. Theorems from Chapter III, Section 3, pp. 158–159 are proved.

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The articles [5], [6], [10], [1], [15], [11], [17], [16], [12], [14], [8], [3], [4], [9], [2], and [13] provide the notation and terminology for this paper.

One can prove the following propositions:

(1) For every complete lattice $L$ and for every net $N$ in $L$ holds $\inf N \leqslant \liminf N$.

(2) Let $L$ be a complete lattice, $N$ be a net in $L$, and $x$ be an element of $L$. Suppose that for every subnet $M$ of $N$ holds $x = \liminf M$. Then $x = \liminf N$ and for every subnet $M$ of $N$ holds $x \geqslant \inf M$.

(3) Let $L$ be a complete lattice, $N$ be a net in $L$, and $x$ be an element of $L$. Suppose $N \in \mathrm{NetUniv}(L)$. Suppose that for every subnet $M$ of $N$ such that $M \in \mathrm{NetUniv}(L)$ holds $x = \liminf M$. Then $x = \liminf N$ and for every subnet $M$ of $N$ such that $M \in \mathrm{NetUniv}(L)$ holds $x \geqslant \inf M$.

Let $N$ be a non empty relational structure and let $f$ be a map from $N$ into $N$. We say that $f$ is greater or equal to id if and only if:

(Def. 1) For every element $j$ of the carrier of $N$ holds $j \leqslant f(j)$.

We now state three propositions:

(4) For every reflexive non empty relational structure $N$ holds $\mathrm{id}_N$ is greater or equal to id.

(5) Let $N$ be a directed non empty relational structure and $x$, $y$ be elements of $N$. Then there exists an element $z$ of $N$ such that $x \leqslant z$ and $y \leqslant z$.

(6) For every directed non empty relational structure $N$ holds there exists a map from $N$ into $N$ which is greater or equal to id.

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Let $N$ be a directed non empty relational structure. One can verify that there exists a map from $N$ into $N$ which is greater or equal to id.

Let $N$ be a reflexive non empty relational structure. Observe that there exists a map from $N$ into $N$ which is greater or equal to id.

Let $L$ be a non empty 1-sorted structure, let $N$ be a non empty net structure over $L$, and let $f$ be a map from $N$ into $N$. The functor $N \cdot f$ yielding a strict non empty net structure over $L$ is defined by the conditions (Def. 2).

(Def. 2)(i) The relational structure of $N \cdot f =$ the relational structure of $N$, and

(ii) the mapping of $N \cdot f =$ (the mapping of $N$) $\cdot f$.

The following propositions are true:

(7) Let $L$ be a non empty 1-sorted structure, $N$ be a non empty net structure over $L$, and $f$ be a map from $N$ into $N$. Then the carrier of $N \cdot f =$ the carrier of $N$.

(8) Let $L$ be a non empty 1-sorted structure, $N$ be a non empty net structure over $L$, and $f$ be a map from $N$ into $N$. Then $N \cdot f = \langle$the carrier of $N$, the internal relation of $N$, (the mapping of $N$) $\cdot f\rangle$.

(9) Let $L$ be a non empty 1-sorted structure, $N$ be a transitive directed non empty relational structure, and $f$ be a function from the carrier of $N$ into the carrier of $L$. Then $\langle$the carrier of $N$, the internal relation of $N$, $f\rangle$ is a net in $L$.

Let $L$ be a non empty 1-sorted structure, let $N$ be a transitive directed non empty relational structure, and let $f$ be a function from the carrier of $N$ into the carrier of $L$. Note that $\langle$the carrier of $N$, the internal relation of $N$, $f\rangle$ is transitive directed and non empty.

We now state the proposition

(10) Let $L$ be a non empty 1-sorted structure, $N$ be a net in $L$, and $p$ be a map from $N$ into $N$. Then $N \cdot p$ is a net in $L$.

Let $L$ be a non empty 1-sorted structure, let $N$ be a net in $L$, and let $p$ be a map from $N$ into $N$. Note that $N \cdot p$ is transitive and directed.

Next we state two propositions:

(11) Let $L$ be a non empty 1-sorted structure, $N$ be a net in $L$, and $p$ be a map from $N$ into $N$. If $N \in \mathrm{NetUniv}(L)$, then $N \cdot p \in \mathrm{NetUniv}(L)$.

(12) Let $L$ be a non empty 1-sorted structure and $N$, $M$ be nets in $L$. Suppose the net structure of $N =$ the net structure of $M$. Then $M$ is a subnet of $N$.

Let $L$ be a non empty 1-sorted structure and let $N$ be a net in $L$. Note that there exists a subnet of $N$ which is strict.

The following proposition is true

(13) Let $L$ be a non empty 1-sorted structure, $N$ be a net in $L$, and $p$ be a greater or equal to id map from $N$ into $N$. Then $N \cdot p$ is a subnet of $N$.

Let $L$ be a non empty 1-sorted structure, let $N$ be a net in $L$, and let $p$ be a greater or equal to id map from $N$ into $N$. Then $N \cdot p$ is a strict subnet of $N$.

One can prove the following two propositions:

(14) Let $L$ be a complete lattice, $N$ be a net in $L$, and $x$ be an element of $L$. Suppose $N \in \mathrm{NetUniv}(L)$. Suppose $x = \liminf N$ and for every subnet $M$ of $N$ such that $M \in \mathrm{NetUniv}(L)$ holds $x \geqslant \inf M$. Then $x = \liminf N$ and for every greater or equal to id map $p$ from $N$ into $N$ holds $x \geqslant \inf(N \cdot p)$.

(15) Let $L$ be a complete lattice, $N$ be a net in $L$, and $x$ be an element of $L$. Suppose $x = \liminf N$ and for every greater or equal to id map $p$ from $N$ into $N$ holds $x \geqslant \inf(N \cdot p)$. Let $M$ be a subnet of $N$. Then $x = \liminf M$.

Let $L$ be a non empty relational structure. The lim inf convergence of $L$ is a convergence class of $L$ and is defined by the condition (Def. 3).

(Def. 3) Let $N$ be a net in $L$. Suppose $N \in \mathrm{NetUniv}(L)$. Let $x$ be an element of the carrier of $L$. Then $\langle N, x\rangle \in$ the lim inf convergence of $L$ if and only if for every subnet $M$ of $N$ holds $x = \liminf M$.

We now state two propositions:

(16) Let $L$ be a complete lattice, $N$ be a net in $L$, and $x$ be an element of $L$. Suppose $N \in \mathrm{NetUniv}(L)$. Then $\langle N, x\rangle \in$ the lim inf convergence of $L$ if and only if for every subnet $M$ of $N$ such that $M \in \mathrm{NetUniv}(L)$ holds $x = \liminf M$.

(17) Let $L$ be a non empty relational structure, $N$ be a constant net in $L$, and $M$ be a subnet of $N$. Then $M$ is constant and the value of $N =$ the value of $M$.

Let $L$ be a non empty relational structure. The functor $\xi(L)$ yielding a family of subsets of $L$ is defined as follows:

(Def. 4) $\xi(L) =$ the topology of ConvergenceSpace(the lim inf convergence of $L$).

The following propositions are true:

(18) For every complete lattice $L$ holds the lim inf convergence of $L$ has (CONSTANTS) property.

(19) For every non empty relational structure $L$ holds the lim inf convergence of $L$ has (SUBNETS) property.

(20) For every continuous complete lattice $L$ holds the lim inf convergence of $L$ has (DIVERGENCE) property.

(21) Let $L$ be a non empty relational structure and $N$, $x$ be sets. If $\langle N, x\rangle \in$ the lim inf convergence of $L$, then $N \in \mathrm{NetUniv}(L)$.

(22) Let $L$ be a non empty 1-sorted structure and $C_1$, $C_2$ be convergence classes of $L$. If $C_1 \subseteq C_2$, then the topology of ConvergenceSpace($C_2$) $\subseteq$ the topology of ConvergenceSpace($C_1$).

(23) Let $L$ be a non empty reflexive relational structure. Then the lim inf convergence of $L \subseteq$ the Scott convergence of $L$.

(24) For all sets $X$, $Y$ such that $X \subseteq Y$ holds $X \in$ the universe of $Y$.

(25) Let $L$ be a non empty transitive reflexive relational structure and $D$ be a directed non empty subset of $L$. Then $\mathrm{NetStr}(D) \in \mathrm{NetUniv}(L)$.

(26) For every complete lattice $L$ and for every directed non empty subset $D$ of $L$ and for every subnet $M$ of $\mathrm{NetStr}(D)$ holds $\liminf M = \sup D$.

(27) Let $L$ be a non empty complete lattice and $D$ be a directed non empty subset of $L$. Then $\langle \mathrm{NetStr}(D), \sup D \rangle \in$ the lim inf convergence of $L$.

(28) For every complete lattice $L$ and for every subset $U_1$ of $L$ such that $U_1 \in \xi(L)$ holds $U_1$ is property(S).

(29) For every non empty reflexive relational structure $L$ and for every subset $A$ of $L$ such that $A \in \sigma(L)$ holds $A \in \xi(L)$.

(30) For every complete lattice $L$ and for every subset $A$ of $L$ such that $A$ is upper holds if $A \in \xi(L)$, then $A \in \sigma(L)$.

(31) Let $L$ be a complete lattice and $A$ be a subset of $L$. Suppose $A$ is lower. Then $-A \in \xi(L)$ if and only if $A$ is closed under directed sups.

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