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The Evaluation of Polynomials

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The articles [11], [15], [12], [3], [2], [17], [4], [18], [1], [13], [14], [9], [6], [7], [19], [16], [20], [5], [8], and [10] provide the terminology and notation for this paper.

1. Preliminaries

The following propositions are true:

(1) For every natural number n holds $0 -' n = 0$.

(2) Let D be a set, p be a finite sequence of elements of D, and i be a natural number. If $i < \operatorname{len} p$, then $p \restriction (i+1) = (p \restriction i) \frown \langle p(i+1) \rangle$.

(3) Let D be a non empty set, p be a finite sequence of elements of D, and n be a natural number. If $1 \leqslant n$ and $n \leqslant \operatorname{len} p$, then $p = (p \restriction (n -' 1)) \frown \langle p(n) \rangle \frown (p_{\restriction n})$.

(4) Let L be an add-associative right zeroed right complementable non empty loop structure and n be a natural number. Then $\sum (n \mapsto 0_L) = 0_L$.

2. About Polynomials

The following propositions are true:

(5) Let L be an add-associative right zeroed right complementable left distributive non empty double loop structure and p be a sequence of L. Then $\mathbf{0}.\, L * p = \mathbf{0}.\, L$.

(6) For every non empty zero structure L holds $\operatorname{len} \mathbf{0}.\, L = 0$.

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(7) For every non degenerated non empty multiplicative loop with zero structure L holds $\operatorname{len} \mathbf{1}. L = 1$.

(8) For every non empty zero structure L and for every Polynomial p of L such that $\operatorname{len} p = 0$ holds $p = \mathbf{0}. L$.

(9) Let L be a right zeroed non empty loop structure, p, q be Polynomials of L, and n be a natural number. If $n \geqslant \operatorname{len} p$ and $n \geqslant \operatorname{len} q$, then $n \geqslant \operatorname{len}(p + q)$.

(10) Let L be an add-associative right zeroed right complementable non empty loop structure and p, q be Polynomials of L. If $\operatorname{len} p \neq \operatorname{len} q$, then $\operatorname{len}(p + q) = \max(\operatorname{len} p, \operatorname{len} q)$.

(11) Let L be an add-associative right zeroed right complementable non empty loop structure and p be a Polynomial of L. Then $\operatorname{len}(-p) = \operatorname{len} p$.

(12) Let L be an add-associative right zeroed right complementable non empty loop structure, p, q be Polynomials of L, and n be a natural number. If $n \geqslant \operatorname{len} p$ and $n \geqslant \operatorname{len} q$, then $n \geqslant \operatorname{len}(p - q)$.

(13) Let L be an add-associative right zeroed right complementable distributive commutative associative left unital field-like non empty double loop structure and p, q be Polynomials of L. If $\operatorname{len} p > 0$ and $\operatorname{len} q > 0$, then $\operatorname{len}(p * q) = (\operatorname{len} p + \operatorname{len} q) - 1$.

3. Leading Monomials

Let L be a non empty zero structure and let p be a Polynomial of L. The functor Leading-Monomial p yielding a sequence of L is defined as follows:

(Def. 1) $(\text{Leading-Monomial}\, p)(\operatorname{len} p -' 1) = p(\operatorname{len} p -' 1)$ and for every natural number n such that $n \neq \operatorname{len} p -' 1$ holds $(\text{Leading-Monomial}\, p)(n) = 0_L$.

The following proposition is true

(14) For every non empty zero structure L and for every Polynomial p of L holds Leading-Monomial $p = \mathbf{0}. L +\cdot (\operatorname{len} p -' 1, p(\operatorname{len} p -' 1))$.

Let L be a non empty zero structure and let p be a Polynomial of L. Observe that Leading-Monomial p is finite-Support.

We now state several propositions:

(15) For every non empty zero structure L and for every Polynomial p of L such that $\operatorname{len} p = 0$ holds Leading-Monomial $p = \mathbf{0}. L$.

(16) For every non empty zero structure L holds Leading-Monomial $\mathbf{0}. L = \mathbf{0}. L$.

(17) For every non degenerated non empty multiplicative loop with zero structure L holds Leading-Monomial $\mathbf{1}. L = \mathbf{1}. L$.

(18) For every non empty zero structure L and for every Polynomial p of L holds $\operatorname{len} \text{Leading-Monomial}\, p = \operatorname{len} p$.

(19) Let L be an add-associative right zeroed right complementable non empty loop structure and p be a Polynomial of L. Suppose $\operatorname{len} p \neq 0$. Then there exists a Polynomial q of L such that $\operatorname{len} q < \operatorname{len} p$ and $p = q + \text{Leading-Monomial}\, p$ and for every natural number n such that $n < \operatorname{len} p - 1$ holds $q(n) = p(n)$.

4. Evaluation of Polynomials

Let L be a unital non empty double loop structure, let p be a Polynomial of L, and let x be an element of the carrier of L. The functor $\operatorname{eval}(p, x)$ yields an element of L and is defined by the condition (Def. 2).

(Def. 2) There exists a finite sequence F of elements of the carrier of L such that $\operatorname{eval}(p, x) = \sum F$ and $\operatorname{len} F = \operatorname{len} p$ and for every natural number n such that $n \in \operatorname{dom} F$ holds $F(n) = p(n -' 1) \cdot \operatorname{power}_L(x, n -' 1)$.

Next we state several propositions:

(20) For every unital non empty double loop structure L and for every element x of the carrier of L holds $\operatorname{eval}(\mathbf{0}. L, x) = 0_L$.

(21) Let L be a well unital add-associative right zeroed right complementable associative non degenerated non empty double loop structure and x be an element of the carrier of L. Then $\operatorname{eval}(\mathbf{1}. L, x) = \mathbf{1}_L$.

(22) Let L be an Abelian add-associative right zeroed right complementable unital left distributive non empty double loop structure, p, q be Polynomials of L, and x be an element of the carrier of L. Then $\operatorname{eval}(p + q, x) = \operatorname{eval}(p, x) + \operatorname{eval}(q, x)$.

(23) Let L be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, p be a Polynomial of L, and x be an element of the carrier of L. Then $\operatorname{eval}(-p, x) = -\operatorname{eval}(p, x)$.

(24) Let L be an Abelian add-associative right zeroed right complementable unital distributive non empty double loop structure, p, q be Polynomials of L, and x be an element of the carrier of L. Then $\operatorname{eval}(p - q, x) = \operatorname{eval}(p, x) - \operatorname{eval}(q, x)$.

(25) Let L be an add-associative right zeroed right complementable right zeroed distributive unital non empty double loop structure, p be a Polynomial of L, and x be an element of the carrier of L. Then $\operatorname{eval}(\text{Leading-Monomial}\, p, x) = p(\operatorname{len} p -' 1) \cdot \operatorname{power}_L(x, \operatorname{len} p -' 1)$.

(26) Let L be an add-associative right zeroed right complementable distributive commutative associative field-like left unital non degenerated non

empty double loop structure, p, q be Polynomials of L, and x be an element of the carrier of L. Then $\mathrm{eval}(\text{Leading-Monomial}\, p * q, x) = \mathrm{eval}(\text{Leading-Monomial}\, p, x) \cdot \mathrm{eval}(q, x)$.

(27) Let L be a field, p, q be Polynomials of L, and x be an element of the carrier of L. Then $\mathrm{eval}(p * q, x) = \mathrm{eval}(p, x) \cdot \mathrm{eval}(q, x)$.

5. Evaluation Homomorphism

Let L be an add-associative right zeroed right complementable distributive unital non empty double loop structure and let x be an element of the carrier of L. The functor Polynom-Evaluation(L, x) yields a map from Polynom-Ring L into L and is defined by:

(Def. 3) For every Polynomial p of L holds (Polynom-Evaluation$(L, x))(p) = \mathrm{eval}(p, x)$.

Let L be an add-associative right zeroed right complementable distributive associative well unital non degenerated non empty double loop structure and let x be an element of the carrier of L. One can verify that Polynom-Evaluation(L, x) is unity-preserving.

Let L be an Abelian add-associative right zeroed right complementable distributive unital non empty double loop structure and let x be an element of the carrier of L. One can verify that Polynom-Evaluation(L, x) is additive.

Let L be a field and let x be an element of the carrier of L. Observe that Polynom-Evaluation(L, x) is multiplicative.

Let L be a field and let x be an element of the carrier of L. Note that Polynom-Evaluation(L, x) is ring homomorphism.

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