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Solving Roots of Polynomial Equations of Degree 2 and 3 with Real Coefficients

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Summary. In this paper, we describe the definition of the first, second, and third degree algebraic equations and their properties. In Section 1, we defined the simple first-degree and second-degree (quadratic) equation and discussed the relation between the roots of each equation and their coefficients. Also, we clarified the form of the root within the range of real numbers. Furthermore, the extraction of the root using the discriminant of equation is clarified. In Section 2, we defined the third-degree (cubic) equation and clarified the relation between the three roots of this equation and its coefficient. Also, the form of these roots for various conditions is discussed. This solution is known as the Cardano solution.

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The terminology and notation used in this paper are introduced in the following articles: [4], [3], [2], [1], [5], and [6].

1. Equation of Degree 1 and 2

Let a, b, x be real numbers. The functor $\mathrm{Poly1}(a, b, x)$ yields a real number and is defined as follows:

(Def. 1) $\mathrm{Poly1}(a, b, x) = a \cdot x + b$.

One can prove the following three propositions:

(1) For all real numbers a, b, x such that $a \neq 0$ holds if $\mathrm{Poly1}(a, b, x) = 0$, then $x = -\frac{b}{a}$.

(2) For every real number x holds $\mathrm{Poly1}(0, 0, x) = 0$.

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(3) For all real numbers a, b, x such that $a = 0$ and $b \neq 0$ it is not true that there exists a real number x such that $\mathrm{Poly1}(a, b, x) = 0$.

Let a, b, c, x be real numbers. The functor $\mathrm{Poly2}(a, b, c, x)$ yields a real number and is defined by:

(Def. 2) $\mathrm{Poly2}(a, b, c, x) = a \cdot x^{\mathbf{2}} + b \cdot x + c$.

One can prove the following propositions:

(4) For all real numbers a, b, c, a', b', c' such that for every real number x holds $\mathrm{Poly2}(a, b, c, x) = \mathrm{Poly2}(a', b', c', x)$ holds $a = a'$ and $b = b'$ and $c = c'$.

(5) Let a, b, c be real numbers. Suppose $a \neq 0$ and $\Delta(a, b, c) \geqslant 0$. Let x be a real number. If $\mathrm{Poly2}(a, b, c, x) = 0$, then $x = \frac{-b+\sqrt{\Delta(a,b,c)}}{2 \cdot a}$ or $x = \frac{-b-\sqrt{\Delta(a,b,c)}}{2 \cdot a}$.

(6) For all real numbers a, b, c, x such that $a \neq 0$ and $\Delta(a, b, c) = 0$ holds if $\mathrm{Poly2}(a, b, c, x) = 0$, then $x = -\frac{b}{2 \cdot a}$.

(7) For all real numbers a, b, c such that $a \neq 0$ and $\Delta(a, b, c) < 0$ it is not true that there exists a real number x such that $\mathrm{Poly2}(a, b, c, x) = 0$.

(8) For all real numbers a, b, c, x such that $a = 0$ and $b \neq 0$ holds if for every real number x holds $\mathrm{Poly2}(a, b, c, x) = 0$, then $x = -\frac{c}{b}$.

(9) For all real numbers a, b, c, x such that $a = 0$ and $b = 0$ and $c = 0$ holds $\mathrm{Poly2}(a, b, c, x) = 0$.

(10) For all real numbers a, b, c such that $a = 0$ and $b = 0$ and $c \neq 0$ it is not true that there exists a real number x such that $\mathrm{Poly2}(a, b, c, x) = 0$.

Let a, x, x_1, x_2 be real numbers. The functor $\mathrm{Quard}(a, x_1, x_2, x)$ yielding a real number is defined by:

(Def. 3) $\mathrm{Quard}(a, x_1, x_2, x) = a \cdot ((x - x_1) \cdot (x - x_2))$.

Next we state the proposition

(11) Let a, b, c, x, x_1, x_2 be real numbers. Suppose $a \neq 0$. Suppose that for every real number x holds $\mathrm{Poly2}(a, b, c, x) = \mathrm{Quard}(a, x_1, x_2, x)$. Then $\frac{b}{a} = -(x_1 + x_2)$ and $\frac{c}{a} = x_1 \cdot x_2$.

2. Equation of Degree 3

Let a, b, c, d, x be real numbers. The functor $\mathrm{Poly3}(a, b, c, d, x)$ yielding a real number is defined as follows:

(Def. 4) $\mathrm{Poly3}(a, b, c, d, x) = a \cdot x^{3}_{\mathbb{N}} + b \cdot x^{\mathbf{2}} + c \cdot x + d$.

Next we state the proposition

(12) Let a, b, c, d, a', b', c', d' be real numbers. Suppose that for every real number x holds $\mathrm{Poly3}(a,b,c,d,x) = \mathrm{Poly3}(a',b',c',d',x)$. Then $a = a'$ and $b = b'$ and $c = c'$ and $d = d'$.

Let a, x, x_1, x_2, x_3 be real numbers. The functor $\mathrm{Tri}(a,x_1,x_2,x_3,x)$ yields a real number and is defined as follows:

(Def. 5) $\mathrm{Tri}(a,x_1,x_2,x_3,x) = a \cdot ((x - x_1) \cdot (x - x_2) \cdot (x - x_3))$.

One can prove the following propositions:

(13) Let a, b, c, d, x, x_1, x_2, x_3 be real numbers. Suppose $a \neq 0$. Suppose that for every real number x holds $\mathrm{Poly3}(a,b,c,d,x) = \mathrm{Tri}(a,x_1,x_2,x_3,x)$. Then $\frac{b}{a} = -(x_1 + x_2 + x_3)$ and $\frac{c}{a} = x_1 \cdot x_2 + x_2 \cdot x_3 + x_1 \cdot x_3$ and $\frac{d}{a} = -x_1 \cdot x_2 \cdot x_3$.

(14) For all real numbers y, h holds $(y+h)^3_{\mathbb{N}} = (y^3_{\mathbb{N}}) + (3 \cdot h \cdot y^{\mathbf{2}} + 3 \cdot h^{\mathbf{2}} \cdot y) + h^3_{\mathbb{N}}$.

(15) Let a, b, c, d, x be real numbers. Suppose $a \neq 0$. Suppose $\mathrm{Poly3}(a,b,c,d,x) = 0$. Let a_1, a_2, a_3, h, y be real numbers. Suppose $y = x + \frac{b}{3 \cdot a}$ and $h = -\frac{b}{3 \cdot a}$ and $a_1 = \frac{b}{a}$ and $a_2 = \frac{c}{a}$ and $a_3 = \frac{d}{a}$. Then $(y^3_{\mathbb{N}}) + ((3 \cdot h + a_1) \cdot y^{\mathbf{2}} + (3 \cdot h^{\mathbf{2}} + 2 \cdot (a_1 \cdot h) + a_2) \cdot y) + ((h^3_{\mathbb{N}}) + a_1 \cdot h^{\mathbf{2}} + (a_2 \cdot h + a_3)) = 0$.

(16) Let a, b, c, d, x be real numbers. Suppose $a \neq 0$. Suppose $\mathrm{Poly3}(a,b,c,d,x) = 0$. Let a_1, a_2, a_3, h, y be real numbers. Suppose $y = x + \frac{b}{3 \cdot a}$ and $h = -\frac{b}{3 \cdot a}$ and $a_1 = \frac{b}{a}$ and $a_2 = \frac{c}{a}$ and $a_3 = \frac{d}{a}$. Then $(y^3_{\mathbb{N}}) + 0 \cdot y^{\mathbf{2}} + \frac{3 \cdot a \cdot c - b^{\mathbf{2}}}{3 \cdot a^{\mathbf{2}}} \cdot y + (2 \cdot (\frac{b}{3 \cdot a})^3_{\mathbb{N}} + \frac{3 \cdot a \cdot d - b \cdot c}{3 \cdot a^{\mathbf{2}}}) = 0$.

(17) Let a, b, c, d, y be real numbers. Suppose $a \neq 0$. Suppose $(y^3_{\mathbb{N}}) + 0 \cdot y^{\mathbf{2}} + \frac{3 \cdot a \cdot c - b^{\mathbf{2}}}{3 \cdot a^{\mathbf{2}}} \cdot y + (2 \cdot (\frac{b}{3 \cdot a})^3_{\mathbb{N}} + \frac{3 \cdot a \cdot d - b \cdot c}{3 \cdot a^{\mathbf{2}}}) = 0$. Let p, q be real numbers. If $p = \frac{3 \cdot a \cdot c - b^{\mathbf{2}}}{3 \cdot a^{\mathbf{2}}}$ and $q = 2 \cdot (\frac{b}{3 \cdot a})^3_{\mathbb{N}} + \frac{3 \cdot a \cdot d - b \cdot c}{3 \cdot a^{\mathbf{2}}}$, then $\mathrm{Poly3}(1,0,p,q,y) = 0$.

(18) Let p, q, y be real numbers. Suppose $\mathrm{Poly3}(1,0,p,q,y) = 0$. Let u, v be real numbers. If $y = u + v$ and $3 \cdot v \cdot u + p = 0$, then $(u^3_{\mathbb{N}}) + v^3_{\mathbb{N}} = -q$ and $(u^3_{\mathbb{N}}) \cdot v^3_{\mathbb{N}} = (-\frac{p}{3})^3_{\mathbb{N}}$.

(19) Let p, q, y be real numbers. Suppose $\mathrm{Poly3}(1,0,p,q,y) = 0$. Let u, v be real numbers. Suppose $y = u + v$ and $3 \cdot v \cdot u + p = 0$. Then

(i) $y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^{\mathbf{2}}}{4} + (\frac{p}{3})^3_{\mathbb{N}}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^{\mathbf{2}}}{4} + (\frac{p}{3})^3_{\mathbb{N}}}}$, or

(ii) $y = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^{\mathbf{2}}}{4} + (\frac{p}{3})^3_{\mathbb{N}}}} + \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^{\mathbf{2}}}{4} + (\frac{p}{3})^3_{\mathbb{N}}}}$, or

(iii) $y = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^{\mathbf{2}}}{4} + (\frac{p}{3})^3_{\mathbb{N}}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^{\mathbf{2}}}{4} + (\frac{p}{3})^3_{\mathbb{N}}}}$.

(20) Let a, b, c, d, x be real numbers. Suppose $a = 0$ and $b \neq 0$ and $\Delta(b,c,d) > 0$. If $\mathrm{Poly3}(a,b,c,d,x) = 0$, then $x = \frac{-c + \sqrt{\Delta(b,c,d)}}{2 \cdot b}$ or $x = \frac{-c - \sqrt{\Delta(b,c,d)}}{2 \cdot b}$.

(21) Let a, b, c, d, p, q, x be real numbers. Suppose $a \neq 0$ and $b = 0$ and $p = \frac{c}{a}$ and $q = \frac{d}{a}$. Suppose $\mathrm{Poly3}(a,b,c,d,x) = 0$. Let u, v be real numbers. Suppose $x = u + v$ and $3 \cdot v \cdot u + p = 0$. Then

(i) $x = \sqrt[3]{-\frac{d}{2\cdot a} + \sqrt{\frac{d^{\mathbf{2}}}{4\cdot a^{\mathbf{2}}} + (\frac{c}{3\cdot a})^3_{\mathbb{N}}}} + \sqrt[3]{-\frac{d}{2\cdot a} - \sqrt{\frac{d^{\mathbf{2}}}{4\cdot a^{\mathbf{2}}} + (\frac{c}{3\cdot a})^3_{\mathbb{N}}}}$, or

(ii) $x = \sqrt[3]{-\frac{d}{2\cdot a} + \sqrt{\frac{d^{\mathbf{2}}}{4\cdot a^{\mathbf{2}}} + (\frac{c}{3\cdot a})^3_{\mathbb{N}}}} + \sqrt[3]{-\frac{d}{2\cdot a} + \sqrt{\frac{d^{\mathbf{2}}}{4\cdot a^{\mathbf{2}}} + (\frac{c}{3\cdot a})^3_{\mathbb{N}}}}$, or

(iii) $x = \sqrt[3]{-\frac{d}{2\cdot a} - \sqrt{\frac{d^{\mathbf{2}}}{4\cdot a^{\mathbf{2}}} + (\frac{c}{3\cdot a})^3_{\mathbb{N}}}} + \sqrt[3]{-\frac{d}{2\cdot a} - \sqrt{\frac{d^{\mathbf{2}}}{4\cdot a^{\mathbf{2}}} + (\frac{c}{3\cdot a})^3_{\mathbb{N}}}}$.

(22) Let a, b, c, d, x be real numbers. Suppose $a \neq 0$ and $\Delta(a, b, c) \geqslant 0$ and $d = 0$. If $\mathrm{Poly3}(a, b, c, d, x) = 0$, then $x = 0$ or $x = \frac{-b+\sqrt{\Delta(a,b,c)}}{2\cdot a}$ or $x = \frac{-b-\sqrt{\Delta(a,b,c)}}{2\cdot a}$.

(23) Let a, b, c, d, x be real numbers. Suppose $a \neq 0$ and $b = 0$ and $\frac{c}{a} < 0$ and $d = 0$. If $\mathrm{Poly3}(a, b, c, d, x) = 0$, then $x = 0$ or $x = \sqrt{-\frac{c}{a}}$ or $x = -\sqrt{-\frac{c}{a}}$.

(24) Let a, b, c, d, x be real numbers. Suppose $a \neq 0$ and $c = 0$. Suppose $\mathrm{Poly3}(a, b, c, d, x) = 0$. Let h be a real number. Suppose $a \cdot x + b = h$ and $h \neq 0$ and $\frac{d}{h} < 0$. Then $x = \frac{h-b}{a}$ or $x = \sqrt{-\frac{d}{h}}$ or $x = -\sqrt{-\frac{d}{h}}$.

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