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The Jónsson Theorem about the Representation of Modular Lattices

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Summary. Formalization of [14, pp. 192–199], chapter IV. Partition Lattices, theorem 8.

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The articles [8], [18], [6], [9], [10], [3], [15], [20], [1], [21], [13], [2], [17], [7], [23], [24], [22], [19], [5], [12], [16], [4], [25], and [11] provide the terminology and notation for this paper.

1. Preliminaries

Let A be a non empty set and let P, R be binary relations on A. Let us observe that $P \subseteq R$ if and only if:

(Def. 1) For all elements a, b of A such that $\langle a, b\rangle \in P$ holds $\langle a, b\rangle \in R$.

Let L be a relational structure. We say that L is finitely typed if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exists a non empty set A such that

(i) for every set e such that $e \in$ the carrier of L holds e is an equivalence relation of A, and

(ii) there exists a natural number o such that for all equivalence relations e_1, e_2 of A and for all sets x, y such that $e_1 \in$ the carrier of L and $e_2 \in$ the carrier of L and $\langle x, y\rangle \in e_1 \sqcup e_2$ there exists a non empty finite sequence F of elements of A such that $\operatorname{len} F = o$ and x and y are joint by F, e_1 and e_2.

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Let L be a lower-bounded lattice and let n be a natural number. We say that L has a representation of type $\leqslant n$ if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exists a non trivial set A and there exists a homomorphism f from L to EqRelPoset(A) such that

(i) f is one-to-one,

(ii) $\mathrm{Im}\, f$ is finitely typed,

(iii) there exists an equivalence relation e of A such that $e \in$ the carrier of $\mathrm{Im}\, f$ and $e \neq \mathrm{id}_A$, and

(iv) the type of $\mathrm{Im}\, f \leqslant n$.

Let us mention that there exists a lattice which is lower-bounded, distributive, and finite.

Let A be a non trivial set. Observe that there exists a non empty sublattice of EqRelPoset(A) which is non trivial, finitely typed, and full.

One can prove the following propositions:

(1) For every non empty set A and for every lower-bounded lattice L and for every distance function d of A, L holds $\mathrm{succ}\, \emptyset \subseteq \mathrm{DistEsti}(d)$.

(2) Every trivial semilattice is modular.

(3) Let A be a non empty set and L be a non empty sublattice of EqRelPoset(A). Then L is trivial or there exists an equivalence relation e of A such that $e \in$ the carrier of L and $e \neq \mathrm{id}_A$.

(4) Let L_1, L_2 be lower-bounded lattices and f be a map from L_1 into L_2. Suppose f is infs-preserving and sups-preserving. Then f is meet-preserving and join-preserving.

(5) For all lower-bounded lattices L_1, L_2 such that L_1 and L_2 are isomorphic and L_1 is modular holds L_2 is modular.

(6) Let S be a lower-bounded non empty poset, T be a non empty poset, and f be a monotone map from S into T. Then $\mathrm{Im}\, f$ is lower-bounded.

(7) Let L be a lower-bounded lattice, x, y be elements of L, A be a non empty set, and f be a homomorphism from L to EqRelPoset(A). If f is one-to-one, then if $f^{\circ}(x) \leqslant f^{\circ}(y)$, then $x \leqslant y$.

2. The Jónsson Theorem

We now state two propositions:

(8) Let A be a non trivial set, L be a finitely typed full non empty sublattice of EqRelPoset(A), and e be an equivalence relation of A. Suppose $e \in$ the carrier of L and $e \neq \mathrm{id}_A$. If the type of $L \leqslant 2$, then L is modular.

(9) For every lower-bounded lattice L such that L has a representation of type $\leqslant 2$ holds L is modular.

Let A be a set. The functor new_set2 A is defined by:

(Def. 4) new_set2 $A = A \cup \{\{A\}, \{\{A\}\}\}$.

Let A be a set. One can verify that new_set2 A is non empty.

Let A be a non empty set, let L be a lower-bounded lattice, let d be a bifunction from A into L, and let q be an element of $[\!: A, A,$ the carrier of $L,$ the carrier of $L :\!]$. The functor new_bi_fun2(d, q) yielding a bifunction from new_set2 A into L is defined by the conditions (Def. 5).

(Def. 5)(i) For all elements u, v of A holds (new_bi_fun2$(d, q))(u, v) = d(u, v)$,

(ii) (new_bi_fun2$(d, q))(\{A\}, \{A\}) = \bot_L$,

(iii) (new_bi_fun2$(d, q))(\{\{A\}\}, \{\{A\}\}) = \bot_L$,

(iv) (new_bi_fun2$(d, q))(\{A\}, \{\{A\}\}) = (d(q_1, q_2) \sqcup q_3) \sqcap q_4$,

(v) (new_bi_fun2$(d, q))(\{\{A\}\}, \{A\}) = (d(q_1, q_2) \sqcup q_3) \sqcap q_4$, and

(vi) for every element u of A holds (new_bi_fun2$(d, q))(u, \{A\}) = d(u, q_1) \sqcup q_3$ and (new_bi_fun2$(d, q))(\{A\}, u) = d(u, q_1) \sqcup q_3$ and (new_bi_fun2$(d, q))(u, \{\{A\}\}) = d(u, q_2) \sqcup q_3$ and (new_bi_fun2$(d, q))(\{\{A\}\}, u) = d(u, q_2) \sqcup q_3$.

Next we state several propositions:

(10) Let A be a non empty set, L be a lower-bounded lattice, and d be a bifunction from A into L. Suppose d is zeroed. Let q be an element of $[\!: A, A,$ the carrier of $L,$ the carrier of $L :\!]$. Then new_bi_fun2(d, q) is zeroed.

(11) Let A be a non empty set, L be a lower-bounded lattice, and d be a bifunction from A into L. Suppose d is symmetric. Let q be an element of $[\!: A, A,$ the carrier of $L,$ the carrier of $L :\!]$. Then new_bi_fun2(d, q) is symmetric.

(12) Let A be a non empty set and L be a lower-bounded lattice. Suppose L is modular. Let d be a bifunction from A into L. Suppose d is symmetric and satisfies triangle inequality. Let q be an element of $[\!: A, A,$ the carrier of $L,$ the carrier of $L :\!]$. If $d(q_1, q_2) \leqslant q_3 \sqcup q_4$, then new_bi_fun2$(d, q)$ satisfies triangle inequality.

(13) For every set A holds $A \subseteq$ new_set2 A.

(14) Let A be a non empty set, L be a lower-bounded lattice, d be a bifunction from A into L, and q be an element of $[\!: A, A,$ the carrier of $L,$ the carrier of $L :\!]$. Then $d \subseteq$ new_bi_fun2(d, q).

Let A be a non empty set and let O be an ordinal number. The functor ConsecutiveSet2(A, O) is defined by the condition (Def. 6).

(Def. 6) There exists a transfinite sequence L_0 such that

(i) ConsecutiveSet2$(A, O) = \operatorname{last} L_0$,

(ii) $\operatorname{dom} L_0 = \operatorname{succ} O$,

(iii) $L_0(\emptyset) = A$,

(iv) for every ordinal number C and for every set z such that $\mathrm{succ}\, C \in \mathrm{succ}\, O$ and $z = L_0(C)$ holds $L_0(\mathrm{succ}\, C) = \mathrm{new\_set2}\, z$, and

(v) for every ordinal number C and for every transfinite sequence L_1 such that $C \in \mathrm{succ}\, O$ and $C \neq \emptyset$ and C is a limit ordinal number and $L_1 = L_0 \restriction C$ holds $L_0(C) = \bigcup \mathrm{rng}\, L_1$.

One can prove the following three propositions:

(15) For every non empty set A holds $\mathrm{ConsecutiveSet2}(A, \emptyset) = A$.

(16) For every non empty set A and for every ordinal number O holds $\mathrm{ConsecutiveSet2}(A, \mathrm{succ}\, O) = \mathrm{new\_set2}\, \mathrm{ConsecutiveSet2}(A, O)$.

(17) Let A be a non empty set, O be an ordinal number, and T be a transfinite sequence. Suppose $O \neq \emptyset$ and O is a limit ordinal number and $\mathrm{dom}\, T = O$ and for every ordinal number O_1 such that $O_1 \in O$ holds $T(O_1) = \mathrm{ConsecutiveSet2}(A, O_1)$. Then $\mathrm{ConsecutiveSet2}(A, O) = \bigcup \mathrm{rng}\, T$.

Let A be a non empty set and let O be an ordinal number. Note that $\mathrm{ConsecutiveSet2}(A, O)$ is non empty.

We now state the proposition

(18) For every non empty set A and for every ordinal number O holds $A \subseteq \mathrm{ConsecutiveSet2}(A, O)$.

Let A be a non empty set, let L be a lower-bounded lattice, let d be a bifunction from A into L, let q be a sequence of quadruples of d, and let O be an ordinal number. Let us assume that $O \in \mathrm{dom}\, q$. The functor $\mathrm{Quadr2}(q, O)$ yielding an element of $[\!:\mathrm{ConsecutiveSet2}(A, O), \mathrm{ConsecutiveSet2}(A, O), \text{the carrier of } L, \text{the carrier of } L:\!]$ is defined by:

(Def. 7) $\mathrm{Quadr2}(q, O) = q(O)$.

Let A be a non empty set, let L be a lower-bounded lattice, let d be a bifunction from A into L, let q be a sequence of quadruples of d, and let O be an ordinal number. The functor $\mathrm{ConsecutiveDelta2}(q, O)$ is defined by the condition (Def. 8).

(Def. 8) There exists a transfinite sequence L_0 such that

(i) $\mathrm{ConsecutiveDelta2}(q, O) = \mathrm{last}\, L_0$,

(ii) $\mathrm{dom}\, L_0 = \mathrm{succ}\, O$,

(iii) $L_0(\emptyset) = d$,

(iv) for every ordinal number C and for every set z such that $\mathrm{succ}\, C \in \mathrm{succ}\, O$ and $z = L_0(C)$ holds $L_0(\mathrm{succ}\, C) = \mathrm{new\_bi\_fun2}(\mathrm{BiFun}(z, \mathrm{ConsecutiveSet2}(A, C), L), \mathrm{Quadr2}(q, C))$, and

(v) for every ordinal number C and for every transfinite sequence L_1 such that $C \in \mathrm{succ}\, O$ and $C \neq \emptyset$ and C is a limit ordinal number and $L_1 = L_0 \restriction C$ holds $L_0(C) = \bigcup \mathrm{rng}\, L_1$.

Next we state several propositions:

(19) Let A be a non empty set, L be a lower-bounded lattice, d be a bifunction from A into L, and q be a sequence of quadruples of d. Then $\mathrm{ConsecutiveDelta2}(q, \emptyset) = d$.

(20) Let A be a non empty set, L be a lower-bounded lattice, d be a bifunction from A into L, q be a sequence of quadruples of d, and O be an ordinal number. Then $\mathrm{ConsecutiveDelta2}(q, \mathrm{succ}\, O) = \mathrm{new\_bi\_fun2}(\mathrm{BiFun}(\mathrm{ConsecutiveDelta2}(q, O), \mathrm{ConsecutiveSet2}(A, O), L), \mathrm{Quadr2}(q, O))$.

(21) Let A be a non empty set, L be a lower-bounded lattice, d be a bifunction from A into L, q be a sequence of quadruples of d, T be a transfinite sequence, and O be an ordinal number. Suppose $O \neq \emptyset$ and O is a limit ordinal number and $\mathrm{dom}\, T = O$ and for every ordinal number O_1 such that $O_1 \in O$ holds $T(O_1) = \mathrm{ConsecutiveDelta2}(q, O_1)$. Then $\mathrm{ConsecutiveDelta2}(q, O) = \bigcup \mathrm{rng}\, T$.

(22) For every non empty set A and for all ordinal numbers O, O_1, O_2 such that $O_1 \subseteq O_2$ holds $\mathrm{ConsecutiveSet2}(A, O_1) \subseteq \mathrm{ConsecutiveSet2}(A, O_2)$.

(23) Let A be a non empty set, L be a lower-bounded lattice, d be a bifunction from A into L, q be a sequence of quadruples of d, and O be an ordinal number. Then $\mathrm{ConsecutiveDelta2}(q, O)$ is a bifunction from $\mathrm{ConsecutiveSet2}(A, O)$ into L.

Let A be a non empty set, let L be a lower-bounded lattice, let d be a bifunction from A into L, let q be a sequence of quadruples of d, and let O be an ordinal number. Then $\mathrm{ConsecutiveDelta2}(q, O)$ is a bifunction from $\mathrm{ConsecutiveSet2}(A, O)$ into L.

The following propositions are true:

(24) Let A be a non empty set, L be a lower-bounded lattice, d be a bifunction from A into L, q be a sequence of quadruples of d, and O be an ordinal number. Then $d \subseteq \mathrm{ConsecutiveDelta2}(q, O)$.

(25) Let A be a non empty set, L be a lower-bounded lattice, d be a bifunction from A into L, O_1, O_2 be ordinal numbers, and q be a sequence of quadruples of d. If $O_1 \subseteq O_2$, then $\mathrm{ConsecutiveDelta2}(q, O_1) \subseteq \mathrm{ConsecutiveDelta2}(q, O_2)$.

(26) Let A be a non empty set, L be a lower-bounded lattice, and d be a bifunction from A into L. Suppose d is zeroed. Let q be a sequence of quadruples of d and O be an ordinal number. Then $\mathrm{ConsecutiveDelta2}(q, O)$ is zeroed.

(27) Let A be a non empty set, L be a lower-bounded lattice, and d be a bifunction from A into L. Suppose d is symmetric. Let q be a sequence of quadruples of d and O be an ordinal number. Then $\mathrm{ConsecutiveDelta2}(q, O)$ is symmetric.

(28) Let A be a non empty set and L be a lower-bounded lattice. Suppose L is modular. Let d be a bifunction from A into L. Suppose d is symmetric and satisfies triangle inequality. Let O be an ordinal number and q be a sequence of quadruples of d. If $O \subseteq \mathrm{DistEsti}(d)$, then $\mathrm{ConsecutiveDelta2}(q, O)$ satisfies triangle inequality.

(29) Let A be a non empty set, L be a lower-bounded modular lattice, d be a distance function of A, L, O be an ordinal number, and q be a sequence of quadruples of d. If $O \subseteq \mathrm{DistEsti}(d)$, then $\mathrm{ConsecutiveDelta2}(q, O)$ is a distance function of $\mathrm{ConsecutiveSet2}(A, O)$, L.

Let A be a non empty set, let L be a lower-bounded lattice, and let d be a bifunction from A into L. The functor $\mathrm{NextSet2}\, d$ is defined by:

(Def. 9) $\mathrm{NextSet2}\, d = \mathrm{ConsecutiveSet2}(A, \mathrm{DistEsti}(d))$.

Let A be a non empty set, let L be a lower-bounded lattice, and let d be a bifunction from A into L. Note that $\mathrm{NextSet2}\, d$ is non empty.

Let A be a non empty set, let L be a lower-bounded lattice, let d be a bifunction from A into L, and let q be a sequence of quadruples of d. The functor $\mathrm{NextDelta2}\, q$ is defined as follows:

(Def. 10) $\mathrm{NextDelta2}\, q = \mathrm{ConsecutiveDelta2}(q, \mathrm{DistEsti}(d))$.

Let A be a non empty set, let L be a lower-bounded modular lattice, let d be a distance function of A, L, and let q be a sequence of quadruples of d. Then $\mathrm{NextDelta2}\, q$ is a distance function of $\mathrm{NextSet2}\, d$, L.

Let A be a non empty set, let L be a lower-bounded lattice, let d be a distance function of A, L, let A_1 be a non empty set, and let d_1 be a distance function of A_1, L. We say that A_1, d_1 is extension2 of A, d if and only if:

(Def. 11) There exists a sequence q of quadruples of d such that $A_1 = \mathrm{NextSet2}\, d$ and $d_1 = \mathrm{NextDelta2}\, q$.

Next we state the proposition

(30) Let A be a non empty set, L be a lower-bounded lattice, d be a distance function of A, L, A_1 be a non empty set, and d_1 be a distance function of A_1, L. Suppose A_1, d_1 is extension2 of A, d. Let x, y be elements of A and a, b be elements of L. Suppose $d(x, y) \leqslant a \sqcup b$. Then there exist elements z_1, z_2 of A_1 such that $d_1(x, z_1) = a$ and $d_1(z_1, z_2) = (d(x, y) \sqcup a) \sqcap b$ and $d_1(z_2, y) = a$.

Let A be a non empty set, let L be a lower-bounded modular lattice, and let d be a distance function of A, L. A function is called a ExtensionSeq2 of A, d if it satisfies the conditions (Def. 12).

(Def. 12)(i) $\mathrm{dom}\, \mathrm{it} = \mathbb{N}$,

(ii) $\mathrm{it}(0) = \langle A, d \rangle$, and

(iii) for every natural number n there exists a non empty set A' and there exists a distance function d' of A', L and there exists a non empty set

A_1 and there exists a distance function d_1 of A_1, L such that A_1, d_1 is extension2 of A', d' and $\mathrm{it}(n) = \langle A', d' \rangle$ and $\mathrm{it}(n+1) = \langle A_1, d_1 \rangle$.

We now state several propositions:

(31) Let A be a non empty set, L be a lower-bounded modular lattice, d be a distance function of A, L, S be a ExtensionSeq2 of A, d, and k, l be natural numbers. If $k \leqslant l$, then $S(k)_{\mathbf{1}} \subseteq S(l)_{\mathbf{1}}$.

(32) Let A be a non empty set, L be a lower-bounded modular lattice, d be a distance function of A, L, S be a ExtensionSeq2 of A, d, and k, l be natural numbers. If $k \leqslant l$, then $S(k)_{\mathbf{2}} \subseteq S(l)_{\mathbf{2}}$.

(33) Let L be a lower-bounded modular lattice, S be a ExtensionSeq2 of the carrier of L, $\delta_0(L)$, and F_1 be a non empty set. Suppose $F_1 = \bigcup\{S(i)_{\mathbf{1}} :$ i ranges over natural numbers$\}$. Then $\bigcup\{S(i)_{\mathbf{2}} :$ i ranges over natural numbers$\}$ is a distance function of F_1, L.

(34) Let L be a lower-bounded modular lattice, S be a ExtensionSeq2 of the carrier of L, $\delta_0(L)$, F_1 be a non empty set, F_2 be a distance function of F_1, L, x, y be elements of F_1, and a, b be elements of L. Suppose $F_1 = \bigcup\{S(i)_{\mathbf{1}} :$ i ranges over natural numbers$\}$ and $F_2 = \bigcup\{S(i)_{\mathbf{2}} :$ i ranges over natural numbers$\}$ and $F_2(x, y) \leqslant a \sqcup b$. Then there exist elements z_1, z_2 of F_1 such that $F_2(x, z_1) = a$ and $F_2(z_1, z_2) = (F_2(x, y) \sqcup a) \sqcap b$ and $F_2(z_2, y) = a$.

(35) Let L be a lower-bounded modular lattice, S be a ExtensionSeq2 of the carrier of L, $\delta_0(L)$, F_1 be a non empty set, F_2 be a distance function of F_1, L, f be a homomorphism from L to $\mathrm{EqRelPoset}(F_1)$, e_1, e_2 be equivalence relations of F_1, and x, y be sets. Suppose that

(i) $f = \alpha(F_2)$,

(ii) $F_1 = \bigcup\{S(i)_{\mathbf{1}} : i$ ranges over natural numbers$\}$,

(iii) $F_2 = \bigcup\{S(i)_{\mathbf{2}} : i$ ranges over natural numbers$\}$,

(iv) $e_1 \in$ the carrier of $\mathrm{Im}\, f$,

(v) $e_2 \in$ the carrier of $\mathrm{Im}\, f$, and

(vi) $\langle x, y \rangle \in e_1 \sqcup e_2$.

Then there exists a non empty finite sequence F of elements of F_1 such that $\mathrm{len}\, F = 2 + 2$ and x and y are joint by F, e_1 and e_2.

(36) For every lower-bounded modular lattice L holds L has a representation of type $\leqslant 2$.

(37) For every lower-bounded lattice L holds L has a representation of type $\leqslant 2$ iff L is modular.

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